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Topological B-model, matrix models, $\hat{c} = 1$ strings and quiver gauge theories

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Abstract: We study topological and integrable aspects of $\hat{c} = 1$ strings. We consider the circle line theories 0A and 0B at particular radii, and the super affine theories at their self-dual radii. We construct their ground rings, identify them with certain quotients of the conifold, and suggest topological B-model descriptions. We consider the partition functions, correlators and Ward identities, and construct a Kontsevich-like matrix model. We then study all these aspects via the topological B-model description. Finally, we analyse the corresponding Dijkgraaf-Vafa type matrix models and quiver gauge theories.

Keywords: Topological Strings, Matrix Models.

Contents

1. Introduction

In this paper we will consider the connections between four types of systems: non-critical strings, topological field theories, matrix models and $\mathcal{N} = 1$ supersymmetric gauge theories in four dimensions (for a recent study see [1]). The most studied example in this context is the non-critical $c = 1$ bosonic string at the self-dual radius. It has been argued to be equivalent to the topological B-model on the deformed conifold, and the corresponding matrix model and gauge theory are described by the $\hat{A}_1$ quiver diagram [2].

A basic ingredient in establishing the connections between these different systems is a commutative and associative ring structure. For the $c = 1$ bosonic string it is the ring of BRST invariant operators with zero dimension and zero ghost number. Its defining relation for vanishing cosmological constant μ is the conifold equation [3]. It has been argued that when $\mu \neq 0$, the defining equation describes the deformed conifold. This ground ring relation has a corresponding chiral ring relation in the gauge theory [2].

The partition function of the $c = 1$ bosonic string as a function of the cosmological constant μ is matched with that of the topological B-model on the deformed conifold, with μ being the deformation parameter, and with the partition function of the $\hat{A}_1$ quiver matrix model, where μ is identified with $g_s N$, N being the size of the matrix. In the four-dimensional gauge theory μ corresponds to the glueball superfield S, and the partition function is identified with the glueball F-terms [2].

Since the $c = 1$ bosonic string is well defined only perturbatively, the above connections are perturbative. It is of interest to ask whether there are other examples of such connections, where the non-critical string is well defined non-perturbatively. Natural candidates are the fermionic $\hat{c} = 1$ strings, which have been introduced recently in [4, 5].

In this paper we will consider two families: the circle line theories 0A and 0B at particular radii, and the super affine theories at their self-dual radii. Our aim is to study aspects of the connections between these non-critical strings and topological field theories, matrix models, and $\mathcal{N} = 1$ supersymmetric gauge theories in four dimensions. We will first construct their ground rings, and identify them with certain Z_2 quotients of the conifold. We will then consider integrable and topological aspects of the theories. We will look at the partition functions, correlators and Ward identities, and construct a 0A Kontsevich-like matrix model description. Finally, we will analyse the corresponding Dijkgraaf-Vafa type matrix models and quiver gauge theories.

The paper is organized as follows. In section 2 we will review some basics of $\hat{c}=1$ strings, and identify candidate radii for a possible topological description. In section 3 we will construct the ground rings and propose topological B-model descriptions. In section 4 we will consider the partition functions and correlators. In section 5 we will discuss the integrable structure and the Kontsevich-like matrix model. In section 6 we will consider the topological B-model description. In section 7 we will analyse the corresponding quiver gauge theories and matrix models. Section 8 is devoted to a discussion.

2. $\hat{c}=1$ strings

Fermionic strings are described by $N=1$ supersymmetric worldsheet field theories coupled to worldsheet supergravity. Consider a nonchiral GSO projection which gives type 0 string theory. There are two distinct choices, depending on how the worldsheet fermion number $(-1)^F$ symmetry is realized in the closed string R-R sector, they are called 0A and 0B. In both theories there are no (NS,R) or (R,NS) sectors, and therefore no spacetime fermions. We will be interested in two-dimensional string theories. Consider the spectrum of these theories. In the NS-NS sector one has the (massless) tachyon physical field, with additional discrete states at special values of the momenta. In the R-R sector there are two vector fields C_1, $\tilde{C}_1$ in type 0A, a scalar C_0 in type 0B theory, and additional discrete states.

Let us compactify the theory on a circle of radius R. We can choose as a matter system a free scalar superfield

$$X = x + i\theta\chi + i\bar{\theta}\bar{\chi} + i\theta\bar{\theta}G\,, \tag{2.1}$$

or any other $\hat{c}=1$ superconformal field theory. We will consider two lines of theories parametrized by the radius R of compactification of the lowest component of X: The 'circle' theories 0A and 0B, and the 'super affine' theories, obtained from the circle by modding out by a Z_2 symmetry $(-)^{\mathbf{F}_L}e^{i\pi p}$ where $(-)^{\mathbf{F}_L}=1$ (-1) on NS-NS (R-R) momentum states, and p is the momentum $k=p/R$ [6].

The vertex operators take the form [5]:

- NS-NS sector (in the $(-1,-1)$ picture):

$$T_k^{(\pm)} = c\bar{c}\exp\left[-(\varphi+\bar{\varphi}) + ik(x_L+x_R) + (1\mp k)\phi\right]. \tag{2.2}$$

- RR sector (in the $(-\frac{1}{2},-\frac{1}{2})$ picture):

$$V_k^{(\pm)} = c\bar{c}\exp\left[-\frac{\varphi+\bar{\varphi}}{2} \mp \frac{i}{2}(H+\bar{H}) + ik(x_L+x_R) + (1\mp k)\phi\right]. \tag{2.3}$$

Winding modes $\hat{T}_k$, $\hat{V}_k$ take the same form with $x_R, \bar{H} \to -x_R, -\bar{H}$, $k=wR/2$. ϕ is the lowest component of the super Liouville field Φ

$$\Phi = \phi + i\theta\psi + i\bar{\theta}\bar{\psi} + i\theta\bar{\theta}F\,, \tag{2.4}$$

and $\psi + i\chi = \sqrt{2}\,e^{iH}$.

Imposing the GSO projection one gets [5]:

- type 0A: In the NS-NS sector we have the tachyon field momentum states with $k = n/R$, and winding states with $k = wR/2$, where here and in the following n, w take integer values. In the R-R sector we remain with the winding modes with $k = wR/2$ and project out the momentum modes.

- type 0B: In the NS-NS sector we have the tachyon field momentum states with $k = n/R$, and winding states with $k = wR/2$. In the R-R sector we remain with the momentum modes with $k = n/R$, while projecting out the winding modes.

- Super affine 0A: In the NS-NS sector we have momentum modes tachyon with $k = 2n/R$ and tachyon winding modes with $k = wR/2$. The R-R states are projected out.

- Superaffine 0B: In the NS-NS sector we have tachyon momentum modes with $k = 2n/R$ and tachyon winding modes with $k = wR/2$. In the R-R sector we have momentum modes with $k = \frac{2n+1}{R}$ and windings with $k = (w + \frac{1}{2})\frac{R}{2}$.

Type 0A and type 0B are T-dual to each other under $R \to \alpha'/R$. The super affine theories are self dual under $R \to 2\alpha'/R$.

The torus partition functions of these theories read [5]:

$$F_{0\mathrm{B}} = -\frac{\ln \mu}{12\sqrt{2}} \left(\frac{R}{\sqrt{\alpha'}} + 2\frac{\sqrt{\alpha'}}{R} \right),$$

$$F_{0\mathrm{A}} = -\frac{\ln \mu}{12\sqrt{2}} \left(2\frac{R}{\sqrt{\alpha'}} + \frac{\sqrt{\alpha'}}{R} \right), \tag{2.5}$$

where μ corresponds to the (renormalized) cosmological constant in the super Liouville action. T-duality interchanges the two. For the superaffine theories we have

$$F_{0\mathrm{A}}^{\text{super-affine}} = -\frac{\ln \mu}{12} \left(\frac{R}{\sqrt{2\alpha'}} + \frac{\sqrt{2\alpha'}}{R} \right),$$

$$F_{0\mathrm{B}}^{\text{super-affine}} = -\frac{\ln \mu}{24} \left(\frac{R}{\sqrt{2\alpha'}} + \frac{\sqrt{2\alpha'}}{R} \right). \tag{2.6}$$

These suggest that we should consider as candidate "topological points" the radii[1]

$$R_{0\mathrm{A}} = \frac{l_s}{\sqrt{2}}, \qquad R_{0\mathrm{B}} = l_s\sqrt{2}, \qquad R_{\text{super-affine}} = l_s\sqrt{2}. \tag{2.7}$$

We will argue that at these points the non-critical $\hat{c} = 1$ string theories have a description as a topological B-model on a Calabi-Yau 3-fold, in a sense that will be made more precise later.

[1]An interesting feature of the radii $R_{0\mathrm{A}}$ and $R_{0\mathrm{B}}$ is that for these values the non-perturbative contributions due to D-instantons and wrapped D0-branes coincide. For a recent discussion of these effects in the CFT and the matrix model approach see [7, 8, 9].

$F_{\mathrm{c=1}}(\mathrm{R_{self-dual}})$	$-\frac{1}{12}\ln\mu$		
$F_{\mathrm{0A}}(\sqrt{\alpha'/2})$	$-\frac{1}{6}\ln\mu$	$F_{\mathrm{0B}}(2\sqrt{\alpha'/2})$	$-\frac{1}{6}\ln\mu$
$F_{\mathrm{0A}}^{\mathrm{super-aff.}}(R_{\text{self-dual}})$	$-\frac{1}{6}\ln\mu$		
$F_{\mathrm{0B}}^{\mathrm{super-aff.}}(R_{\text{self-dual}})$	$-\frac{1}{12}\ln\mu$		

Table 1: Torus partition function at the special radii (2.7).

These radii are chosen in analogy with the $c = 1$ bosonic string at the self-dual radius $R^{\mathrm{c=1}}_{\text{self-dual}} = l_s$, which has a topological description. At these radii the winding modes contribute exactly as the momentum modes to the torus partition function. Note, however, that while the $c = 1$ bosonic string and the superaffine theories are self-dual at $R_{\text{super-affine}} = l_s\sqrt{2}$, type 0A and 0B are not, but are rather T-dual to each other. The results for the torus partition function at the special radii (2.7) are displayed in table 1.

Unless otherwise stated we will employ the conventions $\alpha' = 2$ for the fermionic string and $\alpha' = 1$ for the bosonic string.

3. The ground ring

The spin zero ghost number zero BRST invariant operators generate a commutative, associative ring

$$\mathcal{O}(z)\mathcal{O}'(0) \sim \mathcal{O}''(0) + \{Q, \ldots\}\,, \tag{3.1}$$

called the ground ring, where Q is the BRST operator. Recall that for the $c = 1$ string at the self-dual radius [10], the ground ring is generated by four elements x_{ij}, $i, j = 1, 2$ with the relation [3]

$$x_{11}x_{22} - x_{12}x_{21} = 0\,. \tag{3.2}$$

Note that x_{11}, x_{22} are momentum states, while x_{12}, x_{21} are winding states. Viewed as a complex surface, equation (3.2) describes the conifold.

The ring structure (3.2), which after adding the cosmological constant μ was argued to get deformed to

$$x_{11}x_{22} - x_{12}x_{21} = \mu\,, \tag{3.3}$$

was a first hint that the $c = 1$ string at the self-dual radius is equivalent to a topological model on the deformed conifold geometry (3.3) [11].[2]

In the following we will analyse the ground ring for the circle line theories and the super affine theories at the radii (2.7). We will consider first the theories at $\mu = 0$, and discuss the deformation $\mu \neq 0$ later. The relevant BRST analysis has been performed in [14, 15, 16, 17].

Consider the left sector. The chiral BRST cohomology of dimension zero and ghost number zero is given by the infinite set of states $\Psi_{(r,s)}$ with r, s negative integers

$$\Psi_{(r,s)} \sim O_{r,s}e^{(ik_{r,s}x_L - p_{r,s}\Phi_L)}\,. \tag{3.4}$$

[2] For the topological Landau-Ginzburg description, see [12, 13].

The Liouville and matter momentum are given by

$$k_{r,s} = \frac{1}{2}(r-s), \quad p_{r,s} = \frac{1}{2}(r+s+2)\,. \tag{3.5}$$

The operators $\Psi_{(r,s)}$ are in the NS-sector if $k_{r,s} = (r-s)/2$ takes integer values, and in the R-sector if it takes half integer values.

Of particular relevance for us are the R-sector operators:

$$\begin{aligned} x(z) \equiv \Psi_{(-1,-2)}(z) &= \left(e^{-\frac{i}{2}H} e^{-\frac{1}{2}\varphi} - \frac{1}{\sqrt{2}} e^{\frac{i}{2}H} \partial\xi e^{-\frac{3}{2}\varphi}\right) e^{\frac{i}{2}x - \frac{1}{2}\phi}\,, \\ y(z) \equiv \Psi_{(-2,-1)}(z) &= \left(e^{\frac{i}{2}H} e^{-\frac{1}{2}\varphi} - \frac{1}{\sqrt{2}} e^{-\frac{i}{2}H} \partial\xi e^{-\frac{3}{2}\varphi}\right) e^{-\frac{i}{2}x - \frac{1}{2}\phi}\,, \end{aligned} \tag{3.6}$$

and the NS-sector operators

$$u(z) \equiv \Psi_{(-1,-3)}(z) = x^2\,, \qquad v(z) \equiv \Psi_{(-3,-1)}(z) = y^2\,, \qquad w(z) \equiv \Psi_{(-2,-2)}(z) = xy\,, \tag{3.7}$$

given by

$$\begin{aligned} u(z) &= \left(-e^{-iH} e^{-\varphi} + \frac{i}{\sqrt{2}} c\partial\xi\partial(x - i\phi) e^{-2\varphi} - \sqrt{2} c(\partial^2\xi - \partial\xi\partial\varphi) e^{-2\varphi}\right) e^{ix-\phi}\,, \\ w(z) &= \left(\frac{2\sqrt{2}}{3} i\partial H\, e^{-\varphi} + \frac{i}{3} c\partial\xi \left[\partial(x+i\phi)e^{-iH} - \partial(x-i\phi)e^{iH}\right] e^{-2\varphi} + \right. \\ &\quad \left. + c(\partial^2\xi - \partial\xi\partial\varphi)(e^{iH} + e^{-iH})e^{-2\varphi} - c\partial\xi\partial(e^{iH} + e^{-iH})e^{-2\varphi}\right) e^{-\phi}\,, \\ v(z) &= \left(-e^{iH} e^{-\varphi} + \frac{i}{\sqrt{2}} c\partial\xi\partial(x + i\phi) e^{-2\varphi} + \sqrt{2} c(\partial^2\xi - \partial\xi\partial\varphi) e^{-2\varphi}\right) e^{-ix-\phi}\,. \end{aligned} \tag{3.8}$$

One has the multiplication rule

$$(\Psi_{(r,s)}\Psi_{(r',s')})(z) \sim \Psi_{(r+r'+1,s+s'+1)}(z)\,, \tag{3.9}$$

where $\sim$ indicates that the right hand side could be multiplied by a vanishing constant. The left sector ring of spin zero, ghost number zero, and BRST invariant operators is generated by the elements x and y

$$\Psi_{(r,s)} = x^{-s-1} y^{-r-1}\,, \qquad r,s \in \mathbb{Z}_-\,. \tag{3.10}$$

Similarly, one can construct the ring in the right sector.

In order to construct the ground ring, we combine the left and right sectors with the same left and right Liouville momenta. Denote:

$$a_{ij} = \begin{pmatrix} x\bar{x} & x\bar{y} \\ y\bar{x} & y\bar{y} \end{pmatrix}, \qquad b_{ij} = \begin{pmatrix} u\bar{u} & u\bar{w} & u\bar{v} \\ w\bar{u} & w\bar{w} & w\bar{v} \\ v\bar{u} & v\bar{w} & v\bar{v} \end{pmatrix}. \tag{3.11}$$

Clearly,

$$\det(a_{ij}) = 0\,, \tag{3.12}$$

which is the conifold equation (3.2). The relations among the b_{ij} similarly follow from (3.7). Note that a_{ij} are in the RR sector while b_{ij} are in the NS-NS sector. Note also that a_{12}, a_{21} are winding operators, a_{11}, a_{22} are momentum operators, b_{13}, b_{31} are winding operators and b_{11}, b_{22}, b_{33} momentum operators. $b_{12}, b_{21}, b_{23}, b_{32}$ are mixed winding/momentum operators.

Next we need to impose the GSO projection. We will consider the circle line theories and the superaffine theories at the radii (2.7).

3.1 Circle line theories

Since type 0A and type 0B are T-dual to each other under $R \to \alpha'/R$, we essentially have to consider only one of them.[3] Consider type 0A at the radius $R = l_s/\sqrt{2} = 1$. We impose GSO projection on the operators (3.11) (in the appropriate momenta and winding lattices). In the NS-NS sector we have

$$mm: \ k = n \in \mathbb{Z}\,, \qquad wm: \ k = \frac{w}{2}, w \in \mathbb{Z}\,, \tag{3.13}$$

while in the RR sector only winding modes are allowed,

$$wm: \ k = \frac{w}{2}, w \in \mathbb{Z}\,. \tag{3.14}$$

The ground ring elements that survive the GSO projection are

$$a_{ij} = \begin{pmatrix} & \mathbf{x}\bar{\mathbf{y}} \\ \mathbf{y}\bar{\mathbf{x}} & \end{pmatrix}, \qquad b_{kl} = \begin{pmatrix} \mathbf{u}\bar{\mathbf{u}} & & u\bar{v} \\ & w\bar{w} & \\ v\bar{u} & & \mathbf{v}\bar{\mathbf{v}} \end{pmatrix}, \tag{3.15}$$

where we denoted by bold letters the generators of the ground ring. For instance, b_{22} is generated as $a_{12}a_{21} = b_{22}$, whereas b_{11} cannot be generated from the a_{ij}. Thus, the ground ring at $\mu = 0$ is generated by four elements $a_{12}, a_{21}, b_{11}, b_{33}$ with the relation

$$(a_{12})^2 (a_{21})^2 - b_{11} b_{33} = 0\,. \tag{3.16}$$

The complex 3-fold (3.16) is the Z_2 quotient of the conifold (3.12), with the Z_2 action being

$$\begin{aligned} Z_2: \ (a_{11}, a_{22}) &\to -(a_{11}, a_{22}) \\ (a_{12}, a_{21}) &\to (a_{12}, a_{21})\,. \end{aligned} \tag{3.17}$$

We therefore suggest that *type 0A (0B) at the radius $R = 1$ ($R = 2$) is equivalent to the topological B-model on the (deformed) Z_2 quotient of the conifold (3.17).*

Note that, $c = 1$ bosonic string theory at half the self-dual radius is a Z_2 orbifold of the self-dual radius case [19]. The action on the ground ring elements x_{ij} (3.2) is (3.17), and therefore the invariant ring is as the type 0A one (3.16).

[3] For a discussion of T-duality in finite flux backgrounds see e.g. [18].

Deformation of the ground ring. When the cosmological constant μ and the background RR charge q are nonzero,[4] one expects a deformation of the ground ring relation. Consider first the case $\mu \neq 0$ and $q = 0$. One possibility is to take the Z_2 quotient of the deformed version of (3.12)

$$\det(a_{ij}) = \mu\,. \tag{3.18}$$

This leads to

$$(a_{12}a_{21} + \mu)^2 - b_{11}b_{33} = 0\,. \tag{3.19}$$

Note, that (3.19) is singular at

$$a_{12}a_{21} + \mu = 0, \quad b_{11} = b_{33} = 0\,. \tag{3.20}$$

Matching the Liouville momentum between the deformation and the ground ring relation (3.16) [20], allows another deformed ring relation

$$(a_{12}a_{21} + \mu)(a_{12}a_{21} - \mu) - b_{11}b_{33} = 0\,. \tag{3.21}$$

Note that the complex hypersurface described by (3.21) is non-singular. However, we will argue in section 6 that the deformed ring relation (3.19) is the expected one.

Consider next the case $q \neq 0$. Since the theory has the discrete symmetry $q \to -q$, we expect only even powers of q to appear. Moreover, the Liouville momentum considerations suggest that only a q^2 term will appear in the deformed ring relation. We will argue in section 6, that the expected deformation is

$$(a_{12}a_{21} + \mu)^2 = b_{11}b_{33} - \frac{q^2}{4}\,. \tag{3.22}$$

One can identify two 3-spheres in the geometry (3.22) as follows. First introduce a complex parameter t defined by

$$a_{12}a_{21} = t\,. \tag{3.23}$$

Equation (3.22) can be rewritten as

$$b_{11}b_{33} = \left(t + \mu + \frac{iq}{2}\right)\left(t + \mu - \frac{iq}{2}\right). \tag{3.24}$$

We now regard the geometry of (3.22) as a $\mathbf{C} \times \mathbf{C}$ fibration over the complex plane spanned by t, where the coordinates of the first and second fibration are given by a_{12}, a_{21} and b_{11}, b_{33}, respectively. Each fibration has a topology of the product of two cylinders, whose closed cycles are generated by $(a_{12}, a_{21}) \to (e^{i\theta_1}a_{12}, e^{-i\theta_1}a_{21})$ and $(b_{11}, b_{33}) \to (e^{i\theta_2}b_{11}, e^{-i\theta_2}b_{33})$.

At a generic point of t, the fibration is regular. There are three special points where the fibers get degenerate: at $t = 0$, the first fibration develops a degenerate $\mathbf{S}^1$, while the second is smooth. At $t = -\mu \pm \frac{iq}{2}$, the second fibration degenerates, while the first is regular. To see the two $\mathbf{S}^3$'s, consider

$$C_\pm :\ I_\pm \times \mathbf{S}^1_1 \times \mathbf{S}^1_2\,. \tag{3.25}$$

[4] We will consider a nonzero RR charge of one RR field.

Here $I_\pm$ are the interval on the complex plane between $t = 0$ and $t = -\mu \pm iq/2$. $\mathbf{S}^1_{1,2}$ denote the closed cycles of the first and second cylinders over the intervals. Thus, $C_\pm$ define two $\mathbf{S}^3$'s. Therefore, compared to the deformed conifold analysis [21], we expect two rather than one BPS hypermultiplet from D-brane wrappings in type-II compactification. Using [22]

$$F_1 = -\frac{1}{12}\sum \ln m^2_{BPS}\,, \tag{3.26}$$

we get at $q = 0$[5]

$$F_1 = -2\frac{1}{12}\ln\mu = -\frac{1}{6}\ln\mu\,, \tag{3.27}$$

which agrees with $F_{0A}(\sqrt{\alpha'/2})$.

Note in comparison, that for $c = 1$ bosonic string theory at half the self-dual radius, the one-loop partition function of this theory computed in [23]

$$F_1 = -\frac{1}{24}\left(2 + \frac{1}{2}\right)\ln\mu\,, \tag{3.28}$$

is not an integer multiple of $-\frac{1}{12}\ln\mu$.

3.2 Super affine theories

The super affine theories are self-dual at the radius $R_{\text{super-affine}} = l_s\sqrt{2} = 2$. We consider the ground ring at this radius.

- Super affine 0A: The GSO projection acts on the elements of the ground ring (3.11) by $a_{ij} \to -a_{ij}$ keeping b_{ij} invariant. Therefore the RR operators are projected out, while the NS-NS states remain with

  $$mm:\ k = n \in \mathbb{Z}\,, \qquad wm:\ k = w \in \mathbb{Z}\,. \tag{3.29}$$

  The ground ring is generated by the invariant elements b_{ij} subject to the conditions (3.12) and the projection $a_{ij} \to -a_{ij}$. The complex 3-fold described by the ground ring is the Z_2 quotient of the conifold (3.12)

  $$Z_2:\ a_{ij} \to -a_{ij}\,. \tag{3.30}$$

  We therefore suggest that *super affine 0A at the radius $R = 2$ is equivalent to the topological B-model on the (deformed) Z_2 quotient of the conifold (3.30).*

- Super affine 0B: The GSO projection keeps in (3.11) all the RR operators a_{ij} with

  $$mm:\ k = \frac{n}{2}\quad n \in \mathbb{Z}\,, \qquad wm:\ k = \frac{w}{2}\quad w \in \mathbb{Z}\,, \tag{3.31}$$

  and all the NS-NS operators (b_{ij}) with

  $$mm:\ k = n \in \mathbb{Z}\,, \qquad wm:\ k = \frac{w}{2}\,, \quad w \in \mathbb{Z}\,. \tag{3.32}$$

  The ground ring is generated by the invariant elements a_{ij} subject to the conditions (3.12). The complex 3-fold described by the ground ring is the conifold (3.2). We therefore suggest that *super affine 0B at the radius $R = 2$ is equivalent to the topological B-model on the (deformed) conifold.*

[5] We will discuss the case $q \neq 0$ as well as the higher genus partition functions F_g in section 6.

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Deformation of the ground ring. The singular geometry described by the super affine 0B ground ring at $\mu = 0$ is the conifold (3.2). When $\mu \neq 0$ we expect the ground ring to get deformed in the unique $\mathrm{SU}(2) \times \mathrm{SU}(2)$ invariant way to (3.3).

Recall that the $c = 1$ bosonic string at the self-dual radius is also described by the topological B-model on the deformed conifold. This implies that the two theories are equivalent *perturbatively*. In particular we should have perturbatively

$$\mathcal{F}_{0\mathrm{B}}^{\mathrm{super-aff}}(R_{\text{self-dual}}) = \frac{1}{2}\mu^2 \log\mu - \frac{1}{12}\log\mu + \frac{1}{240}\mu^{-2} + \sum_{g>2} a_g \mu^{2-2g} = \mathcal{F}_{c=1}(R_{\text{self-dual}}) \,. \tag{3.33}$$

$a_g = \frac{B_{2g}}{2g(2g-2)}$ is the Euler class of the moduli space of Riemann surfaces of genus g. Indeed, at genus one, $F_{0\mathrm{B}}^{\mathrm{super-aff.}}(R_{\text{self-dual}}) = F_{c=1}(R_{\text{self-dual}}) = -\frac{1}{12}\ln\mu$.

We expect the super affine 0B at the radius $R = 2$ to provide a *non-perturbative* completion of topological B-model on the deformed conifold.

The singular geometry described by the ground ring of super affine 0A theory at the self-dual radius and $\mu = 0$ is that of a Calabi-Yau space, where a three cycle of the form S^3/Z_2 shrinks to zero size. When $\mu \neq 0$ we expect a deformation of the space such that $Vol(S^3/Z_2) \sim \mu/2$. The Calabi-Yau looks locally like $T^*(S^3/Z_2)$. This is the unique $\mathrm{SU}(2) \times \mathrm{SU}(2)$ invariant deformation (see section 7).

When a three cycle S^3/Γ shrinks to zero size in a Calabi-Yau space, with Γ a freely acting discrete subgroup of $\mathrm{SU}(2)$, then in the deformed background one expects

$$F_1 = -\frac{|\Gamma|}{12}\ln\mu \,. \tag{3.34}$$

$|\Gamma|$ is the order of the group, and (3.34) is derived by counting the number of BPS D-brane wrapping states [24] and using (3.26). When $\Gamma = Z_2$ we get $F_1 = -\frac{1}{6}\ln\mu$, which is in agreement with $F_{0\mathrm{A}}^{\mathrm{super-aff.}}(R_{\text{self-dual}})$.

In the remainder of the paper we will study some topological and integrable aspects of $\hat{c} = 1$ strings. We will analyse partition functions, Ward identities, correlators, the topological B-model and the relation to quiver gauge theories.

4. Matrix models

In this section we will analyse the matrix models of type 0A and superaffine 0A theories on a circle. The type 0A partition function has been computed in [5]. In this section we will compute the superaffine 0A partition function. We will find that *non-perturbatively* at the special radii (2.7)

$$\mathcal{F}_{0\mathrm{A}}^{\mathrm{super-aff.}}(R = \sqrt{2\alpha'}) = \mathcal{F}_{0\mathrm{A}}\left(R = \sqrt{\frac{\alpha'}{2}}\right) , \tag{4.1}$$

when the RR charge $q = 0$. As we will see, this equality seems to work also for the momentum states tachyon correlators, which suggests that superaffine 0A at the self-dual radius

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describes a $(q=0)$ NS-NS sector of 0A at radius $\sqrt{\alpha'/2}$. We will see that *perturbatively*

$$\mathcal{F}_{0\mathrm{A}}^{\mathrm{super-aff}}(R_{\text{self-dual}}) = 2\mathcal{F}_{c=1}(R_{\text{self-dual}})\,, \tag{4.2}$$

and of course the same for $\mathcal{F}_{0\mathrm{A}}(\sqrt{\alpha'/2})$. We will see later that the relations (3.33) and (4.2) appear in other descriptions of the systems such as the quiver $\hat{A}_1$ matrix model and its Z_2 quotients.

These suggest that the (perturbative) partition function of the topological B-models on the Z_2 quotients of the conifold (3.17) and (3.30), as a function of the deformation parameter, is twice that of the $c=1$ bosonic string at the self-dual radius. This is indeed expected from topological considerations [25], namely

$$\mathcal{F}_{S^3/Z_2}(\mu) = 2\mathcal{F}_{S^3}\left(\frac{\mu}{2}\right), \tag{4.3}$$

where $\mathcal{F}_{S^3}$ is the partition function of the topological B-model on the (deformed) conifold and $\mathcal{F}_{S^3/Z_2}$ is the partition function of the topological B-model on the (deformed) Z_2 quotient of the conifold. This follows from (4.2) with the conventions such that $R_{\text{self-dual}} = 1$ in both theories.

4.1 The matrix model analysis

The type 0A matrix model is the theory on the worldvolume of $N+q$ $D0$-branes and N $\bar{D}0$-branes of type 0A string theory. It is a $\mathrm{U}(N)\times\mathrm{U}(N+q)$ theory with a tachyon field described by a complex matrix $t(\tau)$ in the bifundamental [5].

The superaffine 0A matrix model on a circle R is obtained from type 0A theory compactified on a circle with the radius $\mathcal{R}$ by imposing the boundary condition [5]

$$t(\tau+2\pi\mathcal{R}) = t^\dagger(\tau)\,, \tag{4.4}$$

with $\mathcal{R}=R/2$. To see this, recall that the superaffine theory is defined by taking the $\mathbb{Z}_2$ quotient $(-1)^{\mathbf{F}_L}e^{i\pi p}$, where $p\in\mathbb{Z}$ is momentum $k=p/\mathcal{R}$ and $\mathbf{F}_L$ is left space-time fermion number. In the NS-NS sector, the quotient is implemented by using $\mathcal{R}=R/2$, which implies even integer momentum. In the RR sector, the RR one-form potential is odd under the $\mathbb{Z}_2$ action, which exchanges the gauge groups on the D-branes worldvolume $\mathrm{U}(N+q)\leftrightarrow\mathrm{U}(N)$. This is a symmetry only for $q=0$, and is consistent with the fact that the physical spectrum of the superaffine 0A theory does not include RR states.

We expand t as

$$t(\tau) = \sum_n t_n e^{in\tau/(2\mathcal{R})}\,. \tag{4.5}$$

It then follows from (4.4) that t_n obey

$$t_n^\dagger = (-1)^n t_{-n}\,. \tag{4.6}$$

Decompose now $t(\tau)$ as

$$t(\tau) = \frac{1}{\sqrt{2}}(t_+(\tau)+t_-(\tau))\,, \tag{4.7}$$

where

$$t_+(\tau) = \sum_{n=\text{even}} t_n e^{in\tau/(2\mathcal{R})}\,, \qquad t_-(\tau) = \sum_{n=\text{odd}} t_n e^{in\tau/(2\mathcal{R})}\,. \tag{4.8}$$

One finds that

$$(t_+(\tau))^\dagger = t_+(\tau)\,, \qquad (t_-(\tau))^\dagger = -t_-(\tau)\,. \tag{4.9}$$

Hence the matrix model is given by two decoupled (anti-)hermitean matrices

$$\begin{aligned} S &= \beta \int_0^{2\pi\mathcal{R}} d\tau \;\mathrm{tr}\left(\dot{t}^\dagger \dot{t} + \frac{1}{2\alpha'} t^\dagger t\right) \\ &= \frac{\beta}{2}\int_0^{2\pi\mathcal{R}} d\tau \;\mathrm{tr}\left(\dot{t}_+^2 + \frac{1}{2\alpha'}t_+^2 - \dot{t}_-^2 - \frac{1}{2\alpha'}t_-^2\right). \end{aligned} \tag{4.10}$$

Note that

$$t_+(\tau + 2\pi\mathcal{R}) = t_+(\tau)\,, \qquad t_-(\tau + 2\pi\mathcal{R}) = -t_-(\tau)\,. \tag{4.11}$$

We note that the complex matrix model (4.10) is invariant under the $\mathrm{U}(N)_L \times \mathrm{U}(N)_R$ group acting by $t \to U_L\, t\, U_R^\dagger$. With the decomposition of t (4.7), the manifest symmetry is $\mathrm{U}(N)_V \times \mathrm{U}(1)_A$, where $t_\pm$ transform in adjoint representation of $\mathrm{U}(N)_V$, and are in a doublet of $\mathrm{U}(1)_A = O(2)_A$. Since the matrix model is gauged, the Gauss law constraint implies that the physical states are invariant under $\mathrm{U}(N)_V \times \mathrm{U}(1)_A$. To see how the constraint is implemented, consider for simplicity the case $N = 1$. $t_\pm$ yield two independent harmonic oscillators with the creation operators given by $a_\pm^\dagger$. They transform in a doublet of $\mathrm{U}(1)_A$ and are invariant under $\mathrm{U}(1)_V$. It thus follows from the Gauss law constraint that the physical states are generated by gauge invariant operators. This corresponds to the doubling of energies observed in the 0A matrix model [5].

4.2 A brief review

In the following we will briefly review some basic aspects of matrix models, which will be needed for the computation of the super affine 0A partition function (For a review, see for instance [26]). Consider a hermitean matrix model with the action given by

$$S = \int_0^{2\pi R} d\tau\, \beta\, \mathrm{tr}\left(\dot{X}^2 + \frac{1}{2\alpha'}X^2\right), \tag{4.12}$$

and with the boundary condition

$$X(\tau + 2\pi R) = \pm X(\tau)\,. \tag{4.13}$$

The corresponding hamiltonian reads:

$$H = \beta\, \mathrm{tr}\left(\frac{1}{2\beta}p^2 - \frac{1}{4\alpha'}X^2\right) \equiv \beta h\,. \tag{4.14}$$

We decompose the matrix X as

$$X = U \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_N \end{pmatrix} U^\dagger\,, \qquad U \in \mathrm{U}(N)\,. \tag{4.15}$$

Under this decomposition, the path integral measure reads

$$DX = \prod_{i=1}^{N} d\lambda_i d\Omega_{\mathrm{U}(N)} \Delta(\lambda)^2 \,, \tag{4.16}$$

where

$$\Delta(\lambda) = \prod_{i<j} (\lambda_i - \lambda_j) \,. \tag{4.17}$$

The angular part $d\Omega_{\mathrm{U}(N)}$ is not relevant for our purposes and will be omitted in the discussion. Upon quantization, the hamiltonian reads

$$\hat{h} = \mathrm{tr}\left(-\frac{1}{2\beta^2}\frac{\partial^2}{\partial X^2} - \frac{1}{4\alpha'}X^2\right) \sum_{i=1}^{N}\left(-\frac{1}{2\beta^2}\frac{1}{\Delta}\frac{\partial^2}{\partial\lambda_i^2}\Delta - \frac{1}{4\alpha'}\lambda_i^2\right) . \tag{4.18}$$

The Schrödinger equation is

$$\hat{h}\Psi(\lambda) = e\Psi(\lambda) \,. \tag{4.19}$$

Defining the wave function $\psi = \Delta\Psi$, the Schrödinger equation reads

$$\sum_{i=1}^{N}\left[-\frac{1}{2\beta^2}\frac{\partial^2}{\partial\lambda_i^2} - \frac{1}{4\alpha'}\lambda_i^2\right]\psi = e\psi \,. \tag{4.20}$$

Thus, ψ describes a system of N free fermions.

Consider the partition function of the matrix model

$$Z = \int DX\, e^{-S} \,. \tag{4.21}$$

It is easy to show that this is equivalent to

$$Z_\Omega = \mathrm{tr}\left(\hat{\Omega}\, e^{-2\pi R\beta\hat{h}}\right) , \tag{4.22}$$

where $\hat{\Omega} = 1$ for the periodic X. For the anti-periodic X, $\hat{\Omega}$ is a $\mathbb{Z}_2$ operation defined by

$$\hat{\Omega}|X\rangle = |-X\rangle \,, \tag{4.23}$$

where $|X\rangle$ is an eigenstate of the Schrödinger operator $\hat{X}$. Z_Ω can be recast as

$$Z_\Omega = \int de\, e^{-2\pi R\beta e}\rho_\Omega(e) \,, \tag{4.24}$$

where $\rho_\Omega(e)$ is the "twisted" density of states defined by

$$\rho_\Omega(e) = \mathrm{tr}\left(\hat{\Omega}\,\delta(\hat{h} - e)\right) . \tag{4.25}$$

This can be written as

$$\rho_\Omega(e) = \frac{\beta^2}{\pi}\,\mathrm{Im}\int_{-\infty}^{\infty} d\lambda\,\langle\lambda|\hat{\Omega}\frac{1}{-\frac{1}{2}\frac{\partial^2}{\partial\lambda^2} + \frac{1}{2}\beta^2\omega^2\lambda^2 - \beta^2 e - i\epsilon}|\lambda\rangle \,, \tag{4.26}$$

with $\omega^2 = -1/(2\alpha')$. Note that the integration range of λ is $-\infty < \lambda < \infty$. This reflects the fact that both sides of the potential are filled with fermions symmetrically, as in the type 0B matrix model. Using

$$\langle\lambda_f|\frac{1}{-\frac{1}{2}\frac{\partial^2}{\partial\lambda^2}+\frac{1}{2}\omega^2\lambda^2+e-i\epsilon}|\lambda_i\rangle = \int_0^\infty dt\, e^{-et}\sqrt{\frac{\omega}{2\pi\sinh(\omega t)}}\times \\ \times e^{-\omega\left((\lambda_i^2+\lambda_f^2)\cosh(\omega t)-2\lambda_i\lambda_f\right)/(2\sinh(\omega t))}\,, \tag{4.27}$$

one gets for $\hat{\Omega}=1$

$$\rho_+(e) = \frac{\beta^2}{\pi}\,\mathrm{Re}\int_0^\infty dt\,\frac{e^{i\beta^2 et}}{2\sinh\left(\frac{\beta t}{2\sqrt{\alpha}}\right)}\,, \tag{4.28}$$

while for the anti-periodic case

$$\rho_-(e) = \frac{\beta^2}{\pi}\,\mathrm{Re}\int_0^\infty dt\,\frac{e^{i\beta^2 et}}{2\cosh\left(\frac{\beta t}{2\sqrt{\alpha}}\right)}\,. \tag{4.29}$$

Here $\alpha = 2\alpha'$.

In order to evaluate the partition function, it is convenient to introduce a chemical potential such that [23]

$$N = \int de\,\rho(e)\frac{1}{1+e^{2\pi R\beta(e-\mu_F)}}\,. \tag{4.30}$$

Define

$$\begin{aligned} g &= \frac{N}{\beta} \qquad \Delta = 1-g\,,\\ \mu &= \mu_c-\mu_F\,,\\ x &= \mu_c - e\,, \end{aligned} \tag{4.31}$$

where μ_c is the critical point defined by $\mu_c = V(\lambda_c),\ V'(\lambda_c)=0$. In the our case, $\mu_c = 0$. The double scaling limit reads

$$\beta\sim N\to\infty\,,\qquad \beta\mu = \text{finite}\sim\frac{1}{g_s}\,. \tag{4.32}$$

Note that the free energy F is a function of N satisfying

$$\frac{1}{2\pi R\beta}\frac{\partial F}{\partial N} = \mu_F\,. \tag{4.33}$$

To get the free energy as a function of $\mu = -\mu_F$, one performs a Legendre transform [23]:

$$-\mathcal{F} \equiv F - 2\pi R\beta N\mu_F\,. \tag{4.34}$$

One can verify that $\mathcal{F}$ obeys

$$\frac{\partial^2\mathcal{F}}{\partial\mu^2} = 2\pi R\beta^2\frac{\partial\Delta}{\partial\mu}\,, \tag{4.35}$$

and $\mathcal{F}$ can be obtained by integrating this equation.

4.3 Solution of super affine 0A matrix model

Consider now the matrix model (4.10). We will first evaluate the density of states

$$\rho_{\mathrm{S0A}}(e) = \mathrm{tr}(\hat{\Omega}\,\delta(\hat{h} - e))\,, \tag{4.36}$$

where the trace is taken over all the physical states of the two harmonic oscillators obeying the Gauss law constraint. $\hat{\Omega}$ acts only on the Hilbert space of t_- as $\hat{\Omega}|t_-\rangle = |-t_-\rangle$, in order to implement the anti-periodic boundary condition for t_-. $\hat{\Omega}$ acts on the oscillators as

$$\hat{\Omega}(a_-, a_-^\dagger)\hat{\Omega} = -(a_-, a_-^\dagger)\,. \tag{4.37}$$

To see this, recall that $\hat{\Omega}$ acts on the operator $\hat{t}_-$ as $\hat{\Omega}\,\hat{t}_-\hat{\Omega} = -\hat{t}_-$, which is consistent with the relation $\hat{t}_- = (a_- + a_-^\dagger)/\sqrt{2}$. As discussed before, the physical states are in a Hilbert space of a single harmonic oscillator with a doubling of the energy

$$\hat{h}|n\rangle = 2e_n|n\rangle\,, \tag{4.38}$$

e_n being the energy of a single harmonic oscillator with the occupation number n. Note that the physical state $|n\rangle$ is invariant under $\hat{\Omega}$. Thus, the computation of the density of states of superaffine 0A matrix model is equivalent to that of a single hermitean matrix with a doubling of the energy:

$$\begin{aligned}\rho_{\mathrm{S0A}}(e) &= \sum_n \delta(2e_n - e) = \frac{1}{2}\sum_n \delta\left(e_n - \frac{e}{2}\right)\\ &= \frac{\beta^2}{4\pi}\,\mathrm{Re}\int_0^\infty dt\,\frac{e^{\frac{i}{2}\beta^2 et}}{2\sinh\left(\frac{\beta t}{2\sqrt{\alpha}}\right)}\,.\end{aligned} \tag{4.39}$$

This is exactly the result for type 0A matrix model when $q = 0$ [5].

Note that the above computation was done by filling both sides of an upside down potential with a symmetric Fermi level, and as we will see, it agrees with the worldsheet one-loop partition computed in [5]. The symmetric Fermi level ensures that the superaffine 0A theory is stable nonperturbatively.

The boundary condition (4.4) means that the only difference between the 0A and superaffine 0A on a circle is the radius of the circle. The partition function of the superaffine 0A theory on a circle with radius R is identical to the partition function of type 0A on a circle with radius $\mathcal{R} = R/2$. Following the procedures reviewed before, we obtain

$$\frac{\partial^2 \mathcal{F}_{\mathrm{S0A}}}{\partial \mu^2} = \pi R \beta^2 \frac{\partial \Delta_{\mathrm{S0A}}}{\partial \mu}\,, \tag{4.40}$$

with

$$\frac{\partial \Delta_{\mathrm{S0A}}}{\partial \mu} = \frac{\sqrt{\alpha'/2}}{\pi\mu}\,\mathrm{Im}\int_0^\infty dt\, e^{-it}\,\frac{t/(2\beta\mu\sqrt{\alpha'/2})}{\sinh(t/(2\beta\mu\sqrt{\alpha'/2}))}\,\frac{t/(\beta\mu R)}{\sinh(t/(\beta\mu R))}\,. \tag{4.41}$$

This is invariant under the self-duality transformation [5]

$$R \to \frac{2\alpha'}{R}\,, \qquad \beta\mu \to \frac{R}{\sqrt{2\alpha'}}\beta\mu\,. \tag{4.42}$$

An alternative simple way to solve the superaffine 0A matrix model is to decompose the $N \times N$ complex matrix as [27]

$$t = V \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_N \end{pmatrix} W^\dagger \,, \qquad V, W \in \mathrm{U}(N)\,. \tag{4.43}$$

Up to the angular part the path integral measure reads

$$\prod_i d\lambda_i \lambda_i \prod_{i<j} (\lambda_i^2 - \lambda_j^2)^2 \equiv \prod_i d\lambda_i \, J \,, \tag{4.44}$$

at each time of τ. The boundary condition (4.4) implies

$$\lambda_i(\tau + 2\pi\mathcal{R}) = \lambda_i(\tau)\,, \qquad V(\tau + 2\pi\mathcal{R}) = W(\tau)\,. \tag{4.45}$$

The path integral with respect to the angular part gives a volume factor of the gauge group, and we will see that we end up with the deformed matrix model [28], with $q = 0$ and the compactification radius given by $\mathcal{R}$.

It follows from (4.44) that upon quantization, the Schrödinger equation of the superaffine 0A matrix model takes the form

$$\sum_{i=1}^N \left[-\frac{1}{2J} \frac{\partial}{\partial \lambda_i} \left(J \frac{\partial}{\partial \lambda_i} \right) - \frac{1}{4\alpha'} \lambda_i^2 \right] \Psi = E \Psi \,. \tag{4.46}$$

Defining the wavefunction χ as

$$\Psi = \frac{1}{\sqrt{J}} \chi \,, \tag{4.47}$$

the Schrödinger equation can be written as

$$\sum_{i=1}^N \left[-\frac{1}{2} \frac{\partial^2}{\partial \lambda_i^2} - \frac{1}{4\alpha'} \lambda_i^2 + \frac{M}{2\lambda_i^2} \right] \chi = E \chi \,, \tag{4.48}$$

with $M = -1/4$. This is the hamiltonian of the deformed matrix model [28]. It is described by a single hermitean matrix Φ whose eigenvalue is equal to λ_i with the potential given by

$$V(\Phi) = \mathrm{tr} \left(-\frac{1}{4\alpha'} \Phi^2 + \frac{M}{2\Phi^2} \right) . \tag{4.49}$$

Consider $\partial \Delta_{\mathrm{S0A}} / \partial \mu$ in detail. One finds that

$$\frac{\partial \Delta_{\mathrm{S0A}}}{\partial \mu} = \frac{\sqrt{\alpha'/2}}{\pi} \left[-\log \mu + \sum_{g \geq 1} (\beta \mu \sqrt{\alpha})^{-2g} f_g(R) \right] , \tag{4.50}$$

where

$$f_g(R) = (2g-1)! \sum_{n=0}^{g} \frac{(2^{2n}-2)(2^{2(g-n)}-2)}{(2n)!\,(2(g-n))!} B_n B_{g-n} \left(\frac{R}{\sqrt{\alpha}} \right)^{-2n} . \tag{4.51}$$

As an example,

$$f_1(R) = \frac{1}{6}\left(1 + \frac{\alpha}{R^2}\right), \qquad f_2(R) = \frac{1}{6}\left(\frac{7}{10} + \frac{\alpha}{R^2} + \frac{7}{10}\frac{\alpha^2}{R^4}\right). \tag{4.52}$$

It thus follows that

$$\mathcal{F}_{\rm S0A} = \frac{R}{4\sqrt{\alpha}}\left[-(\beta\mu\sqrt{\alpha})^2\,\log\mu - 2\log\mu\, f_1(R) + \sum_{g\geq 2}\frac{(\beta\mu\sqrt{\alpha})^{2-2g}}{(g-1)(2g-1)}\, f_g(R)\right] = \mathcal{F}_{0A}(\mathcal{R})\,. \tag{4.53}$$

The one-loop partition function agrees with the result of [5].

Setting $R=\sqrt{\alpha}$, the self-dual radius, one finds that

$$\mathcal{F}_{\rm S0A} = 2\left[-\frac{1}{2}(\beta\mu)^2\log\mu - \frac{1}{12}\log\mu + \sum_{g\geq 2}\frac{B_g}{2g(2g-2)}\,(\beta\mu)^{2-2g}\right] = 2\mathcal{F}_{c=1}\,, \tag{4.54}$$

being twice the partition function of bosonic $c=1$ string theory at the self-dual radius.

4.4 Tachyon scattering

In the following we will argue that the momentum mode tachyon correlators in superaffine 0A at radius R are identical to the momentum mode tachyon correlators in type 0A ($q=0$) at radius $\mathcal{R}=R/2$. With such relation between the radii there is a one to one map between the tachyon momentum states of the two theories (see section 2).

We consider the tachyon scattering in the matrix model framework. As discussed before, the 0A matrix model is the theory of a gauged complex $N\times N$ matrix t with periodic boundary condition $t(\tau+2\pi\mathcal{R})=t(\tau)$. In order to analyse the tachyon scattering we need first to construct the corresponding operator in the complex matrix model. Here it is convenient to use the equivalence between the 0A matrix model and the deformed matrix model. The tachyon operator with a momentum n in the deformed matrix model is given by [29] (see also appendix A)

$$T_n \sim \int d\tau\, e^{in\tau}\,{\rm tr}\, e^{-l\Phi^2}\,. \tag{4.55}$$

This equals

$$T_n \sim \int d\tau\, e^{in\tau}\,{\rm tr}\, e^{-l\,tt^\dagger}\,, \tag{4.56}$$

since $\Phi, t, t^\dagger$ have the same eigenvalues.

Consider next the momentum mode tachyons in superaffine 0A. We need to use the relation between the radii

$$\mathcal{R} = \frac{R}{2}\,, \tag{4.57}$$

and impose the boundary condition $t(\tau+\pi R)=t^\dagger(\tau)$. As discussed before, this modifies the boundary condition of the angular part (4.45), $V(\tau+\pi R)=W(\tau)$, leaving the eigenvalues periodic. Since the tachyon operator T_n is gauge invariant, this modification is irrelevant for it. Therefore the tachyon operators as well as the matrix model action and gauge invariant measure are the same for type 0A matrix model and the superaffine 0A matrix model at radii (4.57). Thus, we expect the same scattering amplitudes of the momentum mode tachyons in both theories.

5. Integrable structure

The integrable structure of the $c = 1$ bosonic string compactified on a circle has been analysed in [30]. It was found that the euclidean 2D string, perturbed purely by tachyon momentum modes (or by winding modes in the T-dual picture), has the integrable structure of the Toda lattice hierarchy. More precisely, the generating functional of the tachyon momentum modes correlators is a tau function $\tau(t, \bar{t})$ of the Toda lattice hierarchy, with the Toda times $\bar{t}, t$ being the sources of the positive and negative momentum modes, respectively.

The Toda hierarchy is quite general and one has to provide additional information in order to select the correct tau function, which is specified by the so called string equations (see e.g. [31] in the $c = 1$ context). In other words, one has to provide the boundary conditions for the Toda differential equations, which determine the flows in the Toda times $t, \bar{t}$. In the context of non-critical string theory the boundary condition is determined by the unperturbed partition function (see e.g. [32]).

In this section we will study the integrable structure of two-dimensional type 0A string theory at the radius $R = l_s/\sqrt{2}$ (2.7). Recently, the Toda integrable structure in the matrix model formulation of type 0A strings was proved in [33]. We will study the matrix model of the system.

First, we will follow [30] and derive Ward identities for the deformed matrix model giving recursion relations for the tachyon correlators of the 0A string. We will then construct a Kontsevich-like matrix model representation of the generating functional of the tachyon correlators. Both results will have an interpretation in terms of the topological description, which will be discussed in the next section. For a review of matrix models and their integrable structure, see for instance [34, 35]. The integrable structure of the deformed matrix model is investigated in detail in [29, 36, 37].

5.1 Type 0A and the deformed matrix model

As noted in [38, 39], the type 0A matrix model [5] can be recast as the deformed matrix model [28].

The $\mathrm{U}(N+q) \times \mathrm{U}(N)$ type 0A matrix model, which is the theory on the worldvolume of $N+q$ $D0$-branes and N $\bar{D}0$-branes of type 0A string theory, can be described by $N \to \infty$ free fermions moving on the half line $\lambda \geq 0$, whose dynamics is governed by the hamiltonian

$$H = -\frac{1}{2}\frac{d^2}{d\lambda^2} - \frac{1}{2\alpha^2}\lambda^2 + \frac{q^2 - 1/4}{2\lambda^2}\,. \tag{5.1}$$

The eigenvalues λ of the matrix are identified with the open string tachyon field on the unstable D-branes [40], which in type 0A is the tachyon on the unstable $D0 - \bar{D}0$ pair. In the original formulation of [5] the 0A string is described by free fermions moving in $2+2q$ dimensions in an inverted harmonic oscillator potential. The curvature of the potential is identified with the mass of the open string tachyon, which reads

$$m_T^2 = -\frac{1}{2\alpha'}\,. \tag{5.2}$$

Thus, in (5.1) we have the relation between the parameter α and the string length according to

$$\alpha = \sqrt{2\alpha'}\,. \tag{5.3}$$

In the conventions we employ throughout the paper we set $\alpha' = 2$.

The reflection coefficient R_p of the deformed matrix model has been derived in [41]. In our conventions it is, up to a p independent phase, given by

$$R_p = \left(\frac{4}{q^2+4\mu^2-\frac{1}{4}}\right)^p \frac{\Gamma\left(\frac{1}{2}+\frac{q}{2}+p-i\mu\right)}{\Gamma\left(\frac{1}{2}+\frac{q}{2}-p+i\mu\right)}\,, \tag{5.4}$$

which is valid for $q \geq 0$ and $p > 0$. The scattering amplitudes in the deformed matrix model can be obtained by applying the rules of [42]. For instance, the two-point function for the collective field is given by

$$\mathcal{A}_2(p,-p) = \int_0^p dx R_{p-x} R_x^*\,. \tag{5.5}$$

The collective field is related to the string theory tachyon via leg pole factors (see appendix A).

5.2 Ward identities

Consider the deformed matrix model compactified on a circle with radius β. This is equivalent to the free fermions description at temperature $T = 1/2\pi\beta$. The allowed fermion momenta along the time direction are $p_m = 1/\beta(m+1/2)$. The discrete momenta for the bosonic collective field are $p_n = n/\beta$.

As in [30] we will introduce two bosonic fields $\partial\phi^{\mathrm{in/out}}(z) = \sum_k \frac{\alpha^{\mathrm{in/out}}}{z^{k+1}}$ such that

$$\left\langle \prod_i^N V_{n_i/\beta} \prod_i^{N'} V_{-n_i/\beta} \right\rangle = -\frac{(i\mu)^{N+N'}}{\beta} \left\langle \prod_i^N \alpha_{n_i}^{\mathrm{out}} \prod_i^{N'} \alpha_{-n_i}^{\mathrm{in}} \right\rangle. \tag{5.6}$$

The in/out fields are related by the S operator [30].

We introduce the generating functional for the connected tachyon correlators

$$\begin{aligned}\mu^2\mathcal{F} &= \left\langle e^{\sum_{n>0} t_n V_{n/\beta}} e^{\sum_{n>0} \bar{t}_n V_{-n/\beta}} \right\rangle_c \\ &= -\frac{1}{\beta}\langle 0| e^{i\mu\sum_{n>0} t_n\alpha_n} S e^{i\mu\sum_{n>0}\bar{t}_n\alpha_{-n}} |0\rangle_c\,,\end{aligned} \tag{5.7}$$

now written in terms of the modes of a single free boson. Introducing $Z = e^{\mu^2\mathcal{F}}$, and the following shorthand for the coherent states

$$|\bar{t}\rangle \equiv e^{i\mu\sum_{n>0}\bar{t}_n\alpha_{-n}}|0\rangle\,, \tag{5.8}$$

we can write the derivative with respect to the coupling $\bar{t}_n$ as

$$\frac{\partial \mathcal{F}}{\partial \bar{t}_n} = -\frac{i}{\mu} Z^{-1} \langle t| \, \alpha_n S \, |\bar{t}\rangle \, . \tag{5.9}$$

Upon fermionization and applying the S operator, this can be written as

$$\frac{\partial \mathcal{F}}{\partial \bar{t}_n} = -\frac{i}{\mu} Z^{-1} \oint \frac{dw}{w^n} \oint \frac{dz}{z} \left[\sum_{m \in Z} R^*_{p_m} R_{n/\beta - p_m} \left(\frac{w}{z}\right)^m \right] \langle t| \, S\psi(z)\bar{\psi}(w) \, |\bar{t}\rangle \, . \tag{5.10}$$

The product of the reflection coefficients simplifies considerably if we set $\beta = 1$. This is the radius (2.7).

We now have $p_m = \left(m + \frac{1}{2}\right)$, and we get

$$\begin{aligned} R^*_{p_m} R_{n-p_m} &= \left(\frac{4}{q^2 + 4\mu^2 - \frac{1}{4}}\right)^n \left(1 + \frac{q}{2} + m + i\mu - n\right)_n \left(\frac{q}{2} - i\mu - m\right)_n \\ &= \left(\frac{4}{q^2 + 4\mu^2 - \frac{1}{4}}\right)^n (-1)^n \left[-1 - \frac{q}{2} - m - i\mu + n\right]_n \left(\frac{q}{2} - i\mu - m\right)_n \\ &= \left(\frac{4}{q^2 + 4\mu^2 - \frac{1}{4}}\right)^n (-1)^n \left(-i\mu - \frac{q}{2} - m\right)_n \left(-i\mu + \frac{q}{2} - m\right)_n . \end{aligned} \tag{5.11}$$

We used the definitions

$$\Gamma(z + n) = (z)_n \Gamma(z) \, , \tag{5.12}$$

where $(z)_n$ is the Pochhammer symbol defined by

$$(z)_n = z(z+1) \cdots (z+n-1) \, . \tag{5.13}$$

Also,

$$[z]_n = z(z-1) \cdots (z-n+1) \, , \tag{5.14}$$

and we have the identity

$$(z)_n = (-1)^n [-z]_n \, . \tag{5.15}$$

In the sum of (5.10) we can replace the Pochhammer symbols by a differential operator and after partial integration we can transform

$$\left(-i\mu - \frac{q}{2} - m\right)_n \to \left(-i\mu - \frac{q}{2} - z\partial_z\right)_n , \tag{5.16}$$

where the derivative acts on $\psi(z)$ in (5.10). Using the identity

$$(a - z\partial_z)_n = (-1)^n z^n z^a \left(\partial_z\right)^n z^{-a} \, , \tag{5.17}$$

we find

$$\left(-i\mu - \frac{q}{2} - z\partial_z\right)_n \left(-i\mu + \frac{q}{2} - z\partial_z\right)_n = z^{n-\hat{q}} z^{-i\mu} \left(\partial_z\right)^n z^{n+q} \left(\partial_z\right)^n z^{i\mu - \hat{q}} \, , \tag{5.18}$$

where for notational convenience we have introduced the hatted quantity $\hat{q} \equiv q/2$. As in [30], we bosonize $\psi(z)$ into a field $\phi(z)$ and shift the zero mode

$$\phi(z) = \tilde{\phi}(z) + (\hat{q} - i\mu)\log z\,, \tag{5.19}$$

to simplify the action of the operator (5.18). Taking the normal ordered product of the two exponentials, expanding around $z = w$, and taking into account that $\frac{1}{z}\sum_{m\in Z}\left(\frac{w}{z}\right)^m$ acts like a delta function we obtain

$$\begin{aligned}\frac{\partial\mathcal{F}}{\partial\bar{t}_n} &= (-1)^{n+1}\frac{i}{\mu}Z^{-1}\left(\frac{4}{q^2+4\mu^2-\frac{1}{4}}\right)^n\sum_{l=0}^{n}\frac{1}{2n+1-l}\binom{n}{l}[n+q]_l\times\\ &\times\oint dww^{n-l} : e^{-i\mu\varphi(w)}\left(\partial_w\right)^{2n+1-l}e^{i\mu\varphi(w)} : Z\,,\end{aligned} \tag{5.20}$$

where the rescaled operator φ has a representation on the coherent states as

$$\partial\varphi(w) = \frac{1}{w}\left(1+i\frac{\hat{q}}{\mu}\right)+\sum_{k\geq 1}kt_kw^{k-1}-\frac{1}{\mu^2}\sum_{k\geq 1}\partial_{t_k}w^{-k-1}\,. \tag{5.21}$$

The standard generators of the $W_{1+\infty}$ algebra are written in terms of the operators of the form

$$P^{(n)} =: e^{-\tilde{\phi}(z)}\left(\partial_z\right)^n e^{\tilde{\phi}(z)} :, \tag{5.22}$$

and derivatives thereof as [43, 44]

$$W^{(n)}(z) = \sum_{l=0}^{n-1}\frac{(-1)^l}{(n-l)l!}\frac{\left([n-1]_l\right)^2}{[2n-2]_l}\partial_z^l P^{(n-l)}(z)\,. \tag{5.23}$$

If we introduce the operators

$$W^{(k)}(z,q) = \sum_{l=0}^{k-1}\frac{(-1)^l}{(k-l)l!}\left[\frac{k-1}{2}+q\right]_l\partial_z^l P^{(k-l)}(z)\,, \tag{5.24}$$

we can write

$$\begin{aligned}: W_{-n}^{(2n+1)}(q) : Z &= \oint dzz^n : W^{(2n+1)}(z,q) : Z\\ &= \sum_{l=0}^{2n}\frac{(-1)^l}{(2n+1-l)l!}[n+q]_l\oint dzz^n\partial_z^l : P^{(2n+1-l)}(z) : Z\\ &= \sum_{l=0}^{2n}\frac{(-1)^l}{(2n+1-l)l!}[n+q]_l\oint dz(-1)^l\left(\partial_z^l z^n\right) : P^{(2n+1-l)}(z) : Z\\ &= \sum_{l=0}^{n}\frac{[n+q]_l}{(2n+1-l)}\binom{n}{l}\oint dzz^{n-l} : P^{(2n+1-l)}(z) : Z\,.\end{aligned} \tag{5.25}$$

It would be interesting to check the algebra of $W^{(n)}(q)$ and its relation to the standard form of the W_∞ algebra. Note that for $q = 0$, we expect only 1/2 of the W_∞ algebra of the $c = 1$ model [29]. Indeed in this case, we only have odd spin generators in our algebra.

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5.3 A Kontsevich-like matrix model

In this section we will derive a $0+0$ dimensional Kontsevich-like *two* matrix model, which describes type 0A strings at the radius $R = \sqrt{\alpha'/2} = 1$ (2.7).

5.3.1 The strategy

Our derivation will follow the approach taken in [30] and [45] for the $c = 1$ bosonic string at the self-dual radius. The generating functional for the tachyon correlators was expressed as the expectation value of the operator S,

$$Z(t, \bar{t}) = \langle t | S | \bar{t} \rangle \,, \tag{5.26}$$

where the state $|t\rangle$ is a coherent state of a holomorphic boson with the expansion $\partial\varphi(z) = \sum_n \alpha_n z^{-n-1}$ (5.8).

As in the bosonic case, φ may be thought of as the NS-NS tachyon at spatial infinity with periodic euclidean time. We make contact with the two dimensional 0A string theory by writing

$$Z(t, \bar{t}) = \left\langle e^{\sum_{n\geq 1} t_n V_{n/R} + \sum_{n \geq 1} \bar{t}_n V_{-n/R}} \right\rangle , \tag{5.27}$$

where $V_{n/R}$ are the tachyon vertex operators with (euclidean) time compactified at radius R.

The evaluation of (5.26) proceeds by expressing the two-dimensional boson φ using fermions

$$\partial\varphi(z) =: \bar{\psi}(z)\psi(z) : \,. \tag{5.28}$$

In this language the action of the operator S is encoded in the reflection coefficients R_{p_n}

$$S\psi_{-n-1/2}S^{-1} = R_{p_n}\psi_{-n-1/2} \,, \tag{5.29}$$

where $p_n = \frac{1}{R}\left(n + \frac{1}{2}\right)$ is the fermionic momentum along the compact direction. Here, the reflection coefficients R_{p_n} are given by (5.4). It is convenient to represent the fermionic Fock space in terms of semi-infinite forms. The vacuum $|0\rangle$ in this case is written as

$$|0\rangle = z^0 \wedge z^1 \wedge z^2 \wedge \cdots \,. \tag{5.30}$$

The operators $\psi_{n+1/2}, \bar{\psi}_{n+1/2}$ annihilate the vacuum for positive n, and are represented by

$$\psi_{n+1/2} = z^n \,, \qquad \bar{\psi}_{-n-1/2} = \frac{\partial}{\partial z^n} \,. \tag{5.31}$$

Hence, S acts as

$$S : z^n \to R_{-p_n} z^n \,. \tag{5.32}$$

The operator $U(\bar{t})$ that creates the coherent state (5.8) from the vacuum, acts on the semi-infinite forms as

$$U(\bar{t}) : z^n \to e^{i\mu \sum \bar{t}_k z^{-k}} z^n = \sum_{k=0}^{\infty} P_k(i\mu\bar{t}) z^{n-k} \,, \tag{5.33}$$

where $P_k(\bar{t})$ are the Schur polynomials. The combined action of S and $U(\bar{t})$ reads

$$S \circ U(\bar{t}) : z^n \to w^{(n)}(z, \bar{t}) = \sum_{k=0}^{\infty} P_k(i\mu\bar{t}) R_{-p_{n-k}} z^{n-k} \,, \tag{5.34}$$

such that

$$S\,|\bar{t}\rangle = w^{(0)}(z, \bar{t}) \wedge w^{(1)}(z, \bar{t}) \wedge \cdots \,. \tag{5.35}$$

In order to express the bra coherent state $\langle t|$, we need the Miwa transformation, which relates the sources t_n to a constant matrix A by

$$i\mu t_n = -\frac{1}{n} \operatorname{Tr} A^{-n} \,. \tag{5.36}$$

Denoting the eigenvalues of the $N \times N$ matrix A by a_i, $i = 1, \ldots, N$ we can write

$$\langle t| = \langle 0| \prod_{i=1}^{N} e^{-\sum_{n>0} \frac{1}{n} \frac{\alpha_n}{a_i^n}} = \langle N| \frac{\prod_{i=1}^{N} \psi(a_i)}{\Delta(a)} \,, \tag{5.37}$$

where $\Delta(a)$ denotes the Vandermonde determinant and

$$|N\rangle = z^N \wedge z^{N+1} \wedge z^{N+2} \wedge \cdots \tag{5.38}$$

denotes the N-fermion state. We now put together (5.35) and (5.37) and get

$$Z(t, \bar{t}) = \langle t|S|\bar{t}\rangle = \frac{\det w^{(j-1)}(a_i)}{\Delta(a)} \,. \tag{5.39}$$

Thus, the derivation is exactly the same as in the bosonic case, where the distinctive features of the system enter in the reflection coefficients R_{p_n}.

5.3.2 The derivation

Let us compute first when the RR background flux is set to zero $q = 0$. The exact reflection coefficients are given by

$$R_p = K^{p - i\mu} e^{-i2\mu} \frac{\Gamma\left(\frac{1}{2} + p - i\mu\right)}{\Gamma\left(\frac{1}{2} - p + i\mu\right)} \,, \tag{5.40}$$

with

$$K = \frac{4}{4\mu^2 - \frac{1}{4}} \,. \tag{5.41}$$

We will consider only the perturbative properties of the reflection coefficients. Using Gamma function identities, we can rewrite the reflection coefficient R_p as

$$R_p = \frac{K^p}{2\pi} e^{i\pi p} e^{\pi\mu} \left[\Gamma\left(\frac{1}{2} + p - i\mu\right)\right]^2 \,, \tag{5.42}$$

up to terms $O\left(e^{-\mu}\right)$ and a p independent phase.

To calculate the one particle states $w^{(n)}(z,\bar t)$ we plug (5.42) into (5.34) and use the integral representation of the Gamma function. This yields

$$\begin{aligned} w^{(n)}(z,\bar t) &= -\frac{i}{2\pi}e^{-i2\mu}e^{-i\pi n}e^{\pi\mu}K^{-i\mu-n-1/2}z^n \times \\ &\quad\times\int_0^\infty dm_1dm_2\,(m_1m_2)^{-n-i\mu-1}\,e^{-m_1-m_2}e^{i\mu\sum_{k>0}\bar t_k(-K)^k\left(\frac{m_1m_2}{z}\right)^k} \qquad (5.43)\\ &= -\frac{i}{2\pi}e^{-i2\mu}K^{-1/2}z^{-i\mu}\int_0^\infty dm_1dm_2\,(m_1m_2)^{-n-i\mu-1}e^{-m_2}e^{\frac{z}{K}m_1}e^{i\mu\sum_{k>0}\bar t_k(m_1m_2)^k}. \end{aligned}$$

Here we used the relation (5.33), that involves the Schur polynomial. Introduce now two new integration variables by $m=m_1m_2$, $s=m_1$. We have

$$w^{(n)}(z,\bar t) = -\frac{i}{2\pi}e^{-i2\mu}K^{-1/2}z^{-i\mu}\int_0^\infty ds\,s^{-1}e^{\frac{z}{K}s}\int_0^\infty dm\,m^{-n-i\mu-1}e^{-m/s}e^{i\mu\sum_{k>0}\bar t_k m^k}\,. \tag{5.44}$$

Evaluating the expression (5.39), we get

$$\begin{aligned} \frac{\det w^{(j-1)}(a_i,\bar t)}{\Delta(a)} &= c(\mu)^N\prod_i a_i^{-i\mu}\prod_i\left(\int_0^\infty ds_is_i^{-1}e^{\frac{a_i}{K}s_i}\int_0^\infty dm_im_i^{-i\mu-1}e^{-m_is_i^{-1}}e^{i\mu\sum_{k>0}\bar t_km_i^k}\right)\times \\ &\quad\times\frac{\det\left(m_i^{-1}\right)^{j-1}}{\Delta(a)}\,, \qquad (5.45) \end{aligned}$$

where $c(\mu)=-\frac{i}{2\pi}e^{-i2\mu}K^{-1/2}$. Note that

$$\det\left(m_i^{-1}\right)^{j-1} = \Delta(m^{-1}) = \frac{\Delta(m)}{\prod_i m_i^{N-1}}\,. \tag{5.46}$$

To rewrite the integral expression in terms of a matrix integral, we use the Harish-Chandra/ Itzykson-Zuber formula [46, 47]

$$\int d\Omega_{\mathrm{U}(N)}e^{\mathrm{Tr}(\Omega^\dagger x\Omega y)} = \prod_{k=1}^{N-1}k!\frac{\det e^{x_iy_j}}{\Delta(x)\Delta(y)}\,, \tag{5.47}$$

where x,y are diagonal matrices. From this formula, one can verify that for any symmetric function of $g(x_i)$

$$\int[dX]e^{\mathrm{tr}(XY)}g(X)\sim\int\prod_{i=1}^N dx_i\,\frac{\Delta(x)}{\Delta(y)}\,e^{\sum_i x_iy_i}g(x)\,, \tag{5.48}$$

with

$$X=\Omega_{\mathrm{U}(N)}x\,\Omega^\dagger_{\mathrm{U}(N)}\,,\quad Y=UyU^\dagger\qquad\text{for any }U\in\mathrm{U}(N)\,. \tag{5.49}$$

It then follows that

$$Z(t,\bar t)\sim c(\mu)^N\,(\det A)^{-i\mu}\int\prod_{i=1}^N\left(\frac{ds_i}{s_i^N}\,e^{\frac{a_i}{K}s_i}\right)\frac{\Delta(s)}{\Delta(a)}\times$$

$$
\begin{aligned}
&\times \int [dM]\,(\det M)^{-i\mu-N}\, e^{i\mu\sum_k \bar{t}_k \operatorname{tr} M^k - \operatorname{tr}(MS^{-1})} \\
&\sim c(\mu)^N K^{-\frac{N(N-1)}{2}} (\det A)^{-i\mu} \int [dM_1]\, e^{\frac{1}{K}\operatorname{Tr}(M_1 A) - N \operatorname{tr}\log M_1} \times \\
&\times \int [dM_2]\, e^{-\operatorname{Tr}\left(M_1^{-1} M_2\right)} e^{-(i\mu+N)\operatorname{Tr}\log M_2} e^{i\mu\sum_{k>0}\bar{t}_k \operatorname{Tr} M_2^k}\,,
\end{aligned}
\tag{5.50}
$$

where by $\sim$ we mean up to a constant factor. This is a Kontsevich-Penner like *two* matrix model over positive definite matrices, which describes type 0A strings at the radius $R = 1$. The convergence properties of this integral are similar to those of the Imbimbo-Mukhi matrix integral [45], which derive directly from the properties of the Γ-functions.

Consider next the case of non vanishing RR background flux $q \neq 0$. Instead of (5.40), we use now the reflection coefficients for finite RR background flux (5.4), which we can rewrite in perturbation theory as

$$
R_p = \frac{\tilde{K}^{p-i\mu}}{2\pi} e^{i\pi p - i2\mu} e^{\pi\mu} \Gamma\left(\frac{1}{2} + \frac{q}{2} + p - i\mu\right) \Gamma\left(\frac{1}{2} - \frac{q}{2} + p - i\mu\right). \tag{5.51}
$$

Following the same steps as above, we find that we can write the generating functional of the momentum mode tachyon correlators as

$$
\begin{aligned}
Z(t,\bar{t}) &\sim \tilde{c}(\mu)^N \tilde{K}^{-\frac{N(N-1)}{2}} (\det A)^{\frac{q}{2}-i\mu} \int [dM_1]\, e^{\frac{1}{K}\operatorname{Tr}(M_1 A)} e^{-(N-q)\operatorname{Tr}\log M_1} \times \\
&\times \int [dM_2]\, e^{-\operatorname{Tr}\left(M_1^{-1} M_2\right)} e^{-(q/2+i\mu+N)\operatorname{Tr}\log M_2} e^{i\mu\sum_{k>0}\bar{t}_k \operatorname{Tr} M_2^k}\,,
\end{aligned}
\tag{5.52}
$$

with $\tilde{c}(\mu) = \frac{i}{2\pi}(-1)^{q/2+1} e^{-i2\mu} \tilde{K}^{q/2-1/2}$, and $\tilde{K} = \frac{4}{q^2+4\mu^2-1/4}$. It is obvious that (5.52) reproduces (5.50) in the case $q = 0$. It would be interesting to generalize the matrix model to other radii by using a normal matrix model as was done for the $c = 1$ string in [48].

5.3.3 The partition function

We now set $\bar{t}$ to zero in order to get the unperturbed partition function. Consider the case $q = 0$. We can write Z as a product of two independent matrix integrals

$$
\begin{aligned}
Z(t=\bar{t}=0) &\sim \mathcal{A}(\mu)\mathcal{N}^{-2} \int [dM_1]\, e^{\frac{1}{K}\operatorname{Tr}(M_1) - (i\mu+N)\operatorname{Tr}\log M_1} \times \\
&\times \int [dM_2]\, e^{-\operatorname{Tr}(M_2) - (i\mu+N)\operatorname{Tr}\log M_2}\,,
\end{aligned}
\tag{5.53}
$$

with $\mathcal{A}(\mu) \sim K^{i\mu N + \frac{N^2}{2}}, \mathcal{N} = \int dM e^{-N \operatorname{tr}\frac{1}{2}M^2}$. After rescaling and shifting the integration variables, Z is written as a square of the Penner integral

$$
Z \sim \mathcal{N}^{-2}\left(\int dM e^{-Nt\sum_{k>2}\operatorname{Tr}\frac{M^k}{k}}\right)^2\,, \tag{5.54}
$$

where $t = -(1 + i\mu/N)$. In the limit of large N, we reproduce the partition function of the 0A theory compactified at $R = 1$ in units of $\alpha' = 2$ [5].

6. Topological B-model

The topological B-model is a twisted $N = 2$ supersymmetric σ-model. Its marginal deformations are complex deformations of the target space [49]. Thus, the topological B-model studies the complex deformations of the target space, which is known as the Kodaira Spencer theory [50]. The basic field of the B-model $A \in H^{(0,1)}(T^{1,0}M)$ is called the Kodaira-Spencer Field.

Recently, a simplified framework for studying the topological B-model on non-compact Calabi-Yau manifolds M of the form

$$uw - H(x, y) = 0\,, \tag{6.1}$$

has been suggested in [1]. In this section we would like to consider some aspects of $\hat{c} = 1$ strings via a topological description in this framework.

The ground ring of superaffine 0B at the self-dual radius takes the form (6.1) with $H(x, y) = xy - \mu$. This case has been studied in detail in [1] in the context of the $c = 1$ bosonic string at the self-dual radius. On the other hand the ground ring of superaffine 0A at the self-dual radius does not take the form (6.1), and we cannot study it in this simplified setting. The (singular) ground rings of type 0A (0B) strings at the radii (2.7) take the form (6.1) and we will consider this case. In addition, we will consider for comparison the Z_n orbifolds of the bosonic $c = 1$ string.

The observables correspond to variations of the complex structure at infinity, and will be identified with the momentum modes of the tachyon field. In fact, one considers only deformations of $H(x, y)$, keeping the dependence on u, w fixed. One can think of the deformations of the Riemann surface $H(x, y) = 0$ as corresponding to the deformations of the Fermi surface in the matrix model by the tachyon perturbations. In this case, the problem reduces to a one complex-dimensional one

$$\int_{\text{3-cycle}} \Omega \to \int_{\text{1-cycle}} y(x)dx\,. \tag{6.2}$$

The complex structure deformations of $H(x, y)$ are encoded in the one-form

$$\lambda = y\, dx\,. \tag{6.3}$$

In each (asymptotic) patch the variations of the complex structure are described by the modes of a chiral boson $\phi(x)$, defined by

$$y(x) = \partial\phi(x) = y_{cl}(x) + t_0 x^{-1} + \sum_{n>0} n t_n x^{n-1} + \sum_{n>0} \mathcal{F}_n x^{-n-1}\,. \tag{6.4}$$

$\phi(x)$ is the Kodaira-Spencer field, and $y_i^{\text{cl}}(x_i)$ is the solution of $H(x, y) = 0$. We will associate the commutator of x and y with the string equation.

One defines a fermion field

$$\psi(x) = e^{\frac{\phi(x_i)}{g_s}}\,. \tag{6.5}$$

As outlined in [1] non-compact branes fibered over the Riemann surface, whose backreaction changes the complex structure, can be described by these essentially free fermions, which, however, do not transform geometrically from patch to patch.[6] Rather it was proposed in [1] that they transform according to

$$\widetilde{\psi}(\widetilde{x}) = \int dx\, e^{-S(x,\widetilde{x})/g_s}\, \psi(x)\,, \tag{6.6}$$

with $S(x,\widetilde{x})$ a canonical transformation relating the $x/\tilde{x}$ patches. For instance, for the $c=1$ string at the self-dual radius, one has the familiar Fourier transformation with [1]

$$S(x,\widetilde{x}) = x\widetilde{x}\,. \tag{6.7}$$

6.1 Partition function and geometry of type 0A strings

In this section we will study the Calabi-Yau B-model geometry corresponding to the type 0A strings at the radius (2.7). We will see how the B-model reproduces precisely the partition function of the type 0A strings, even when the RR flux q is different than zero. We will start by considering for comparison, the Z_n orbifolds of the $c=1$ bosonic string [25], and consider next the type 0A strings. Previous work on the connection between the deformed matrix model and topological field theory appeared in [54].

6.1.1 Z_n orbifolds of the $c=1$ bosonic string

In [25] it was argued that the B-model geometry corresponding to the $c=1$ bosonic string at a radius $R = R_{sd}/n$, with n integer and R_{sd} the self dual radius should arise from an orbifold of the conifold in the following way. The Z_n orbifold group acts on the $c=1$ ground ring elements x_{ij} (3.2) as

$$\{x_{11}, x_{22}, x_{12}, x_{21}\} \to \left\{\omega x_{11}, \omega^{-1} x_{22}, x_{12}, x_{21}\right\}, \tag{6.8}$$

where $\omega^n = 1$ and x_{11}, x_{22} are the momentum modes.

The deformed $c=1$ ring relation (3.3) can be written as

$$x_{11}x_{22} = t\,, \qquad t = x_{12}x_{21} + \mu\,. \tag{6.9}$$

Modding out by Z_n leads to an A_{n-1} singularity, which in terms of the invariant coordinates $w = x_{11}^n, z = x_{22}^n$ is

$$wz = t^n\,, \qquad t = x_{12}x_{21} + \mu\,. \tag{6.10}$$

If we deform the A_{n-1} singularity we get a smooth geometry

$$wz = \prod_i (x_{12}x_{21} - \hat{\mu}_i)\,, \tag{6.11}$$

with $\hat{\mu}_i = \mu_i - \mu$.

We have n inequivalent 3-cycles in this geometry, whose sizes are determined by the parameters $\hat{\mu}_i$. It was suggested in [25], that the partition function of the topological B-model on this geometry, describing the nearly massless hypermultiplets arising from

[6] A similar geometrical interpretation of D-branes in the non-critical string has been recently given in [51] (see also [52, 53]).

D-branes wrapping the n shrinking cycles, should map to the partition function of the Z_n orbifold of the $c = 1$ bosonic string for appropriate $\hat{\mu}_i$. Each of the cycles gives a contribution equal to the conifold one, such that the partition function $\mathcal{F}(\mu)$ of the Z_n quotient can be written as

$$\mathcal{F}(\mu) = \sum_i \mathcal{F}_{c=1}(\hat{\mu}_i)\,, \tag{6.12}$$

where $\mathcal{F}_{c=1}(\hat{\mu}_i)$ is the partition function of the $c = 1$ string at the self dual radius.

Indeed, we can write the free energy of the $c = 1$ string at radius R as (see e.g. [55])

$$\mathcal{F}(\mu, R) = \int_{-\infty}^{\infty} dE \rho(E) \log\left[1 + e^{-\beta(\mu+E)}\right], \tag{6.13}$$

where $\beta = 2\pi R$. If we use the fact that the density of states $\rho(E)$ is related to the reflection coefficient $R(E)$ by

$$\rho(E) \sim \frac{d\phi(E)}{dE}\,, \qquad \phi(E) \sim \log R(E)\,, \tag{6.14}$$

we can integrate by parts and pick up the residues in the upper half plane in (6.13) to obtain

$$\mathcal{F}(\mu, R) = -i \sum_{r \in Z_+ + 1/2} \phi\left(\frac{ir}{R} - \mu\right). \tag{6.15}$$

Plugging the reflection coefficients $R(E) \sim \Gamma(iE + 1/2)$ for the $c = 1$ string we arrive at

$$\mathcal{F}(\mu, R) = \sum_{r,s \in Z_+ + 1/2} \log\left(s + \frac{r}{R} - i\mu\right). \tag{6.16}$$

As shown in [25] for $R = 1/n$ (note that our conventions are $\alpha' = 1$), this can be recast in the form

$$\mathcal{F}(\mu, R) = \sum_{k=-\frac{n-1}{2}}^{\frac{n-1}{2}} \mathcal{F}_{c=1}(\hat{\mu}_k)\,, \tag{6.17}$$

with $\hat{\mu}_k = \frac{\mu + ik}{n}$, thus fixing the deformation parameters of the geometry (6.11).

6.1.2 The type 0A string

We now apply the above arguments to the case of the 0A string. For our current purposes it will be sufficient to know that the perturbative reflection coefficients of the 0A string (5.51) can be written as

$$R(E) \sim \Gamma^2\left(\frac{iE}{2}\alpha + \frac{q+1}{2}\right), \qquad \alpha = \sqrt{2\alpha'}\,. \tag{6.18}$$

In our conventions of $\alpha' = 2$ for the fermionic string, this gives for the partition function

$$\mathcal{F}(\mu, R = 1) = 2\,\mathrm{Re} \sum_{r,s \in Z_+ + 1/2} \log\left(r + s - i\mu + \frac{q}{2}\right), \tag{6.19}$$

which upon comparison to the conifold partition function leads to the perturbative identification

$$\mathcal{F}(\mu, R = 1) = \mathcal{F}_{c=1}\left(\mu + \frac{iq}{2}\right) + \mathcal{F}_{c=1}\left(\mu - \frac{iq}{2}\right). \tag{6.20}$$

The corresponding geometry is (3.22).

Note that in (6.20) we have chosen a linear combination that respects the discrete symmetry $q \to -q$ of the hamiltonian (5.1), naturally leading to a geometry (3.22) with the same symmetry. Furthermore, setting $q = 1/2$, which, of course, is unphysical in the 0A string, and working with the conventions of $\alpha' = 1/2$ we can recover the perturbative partition function of the Z_2 orbifold of the $c = 1$ string (with $\alpha' = 1$). This is expected since for $q = 1/2$ the deformation parameter of the deformed matrix model (5.1) vanishes, and the different curvature of the inverted harmonic oscillator in the case of the fermionic string translates into the appropriate α' conventions.

The geometry (3.22) is consistent with the undeformed ground ring of the 0A string (3.16), and gives a prediction for the deformation of the ground ring induced by the cosmological constant μ and the RR charge q.

In the geometry (3.22) we have two independent parameters μ and q describing deformations of the complex structure. In the neighbourhood of a singularity in the complex structure moduli space the B-model on (3.22) is dominated by a universal behaviour determined by the singularity. We suggest that different perturbative expansions of the full non-perturbative solution of the 0A string describe the B-model on (3.22) near different singularities in the complex structure moduli space, where generally different 3-cycles shrink to zero size. For instance, the perturbative expansion in μ (q) (for the asymptotic expansion in q see e.g. [28, 29] and [56] for a recent discussion), which comprises the terms that diverge in the limit $\mu \to 0$ ($q \to 0$), should describe the (perturbative) B-model near the points $\mu \to 0$ ($q \to 0$) in the complex structure moduli space. In the following, we will discuss the point $\mu \to 0$ with $q = 0$ held fixed. We will be able to recover the integrable structure that describes the 0A tachyon scattering from the topological B-model.

6.2 Reduction of the 3-form

As discussed above, the tachyons of the type 0A string are associated with the complex deformations of the Riemann surface $H(x, y) = 0$. A crucial ingredient of this correspondence is the reduction of the Calabi-Yau geometry to the Riemann surface (6.2). In this section we will calculate the effective one-form on the Riemann surface (6.3). In the $0+1$-dimensional matrix model description, it is spanned by the momentum modes of the ground ring. We will derive the canonical transformation S, which relates the two asymptotic patches of the Riemann surface.

We will first consider the B-model on the geometry (3.22) for $q = 0$. For simplicity we will denote $u = a_{12}$, $v = a_{21}$, $x = b_{11}$, $y = b_{33}$. Thus,

$$(uv + \mu)^2 = xy\,. \tag{6.21}$$

The holomorphic 3-form Ω takes the form

$$\Omega = \frac{dx}{x} \wedge du \wedge dv\,. \tag{6.22}$$

We consider the integral of the three-form Ω over the three-cycles of the form S^1 in the u, v plane, $u \to e^{i\theta}u$, $v \to e^{-i\theta}v$, fibered over a two real dimensional disc in the x, y plane bounded by a 1-cycle on the Riemann surface

$$xy - \mu^2 = 0\,. \tag{6.23}$$

Over the Riemann surface the fiber develops a node at $u = v = 0$, similar to the case of $c = 1$, but there is also a disconnected component that remains regular.

Since the Riemann surface is a subset of the x, y plane we replace

$$du \sim \frac{x dy}{2v\,(uv + \mu)}\,, \tag{6.24}$$

and the 3-form may be written as

$$\Omega = \frac{1}{2} dx \wedge \frac{dy}{\sqrt{xy}} \wedge \frac{dv}{v}\,. \tag{6.25}$$

Integrating over the circle in the fiber gives an effective 2-form in the x, y plane,

$$\int_{fiber} \Omega \to \frac{dx \wedge dy}{\sqrt{xy}}\,. \tag{6.26}$$

The integral over the disk in the x, y plane can be rewritten by using Stokes theorem as an integral of a one-form over the boundary

$$\int \Omega \to \int_{\partial D} \frac{\sqrt{xy}}{y} dy\,, \tag{6.27}$$

such that we remain with an integral over a 1-cycle on the Riemann surface $xy = \mu^2$,

$$\int \Omega \to \oint \mu \frac{dy}{y}\,. \tag{6.28}$$

It is instructive to perform the reduction in a slightly different way. We rewrite the geometry as

$$t = uv + \mu\,, \qquad t^2 = xy\,. \tag{6.29}$$

In these coordinates, the three-cycle is given by two circles in the u, v and x, y fibers, respectively, fibered over an interval $t \in [0, \mu]$ in the t plane. Reducing the 3-form

$$\Omega = \frac{dx}{x} \wedge du \wedge dv = \frac{dx}{x} \wedge \frac{du}{u} \wedge dt\,, \tag{6.30}$$

to a 1-form in the x, y fiber gives

$$\int \Omega \to \int \mu \frac{dx}{x}\,, \tag{6.31}$$

Note that if we change $x \to y$ we get an additional minus sign. This gives the one-form in the second asymptotic patch parametrized by y. In addition, the integral gives the deformation parameters μ as expected. The integration contour is a circle around the origin in the complex y-plane. We might think of this contour as corresponding to the compact one-cycle on the Riemann surface giving rise to the compact three-cycle in the full Calabi-Yau.[7]

[7] There is also a non-compact one cycle corresponding to a different real section of the Riemann surface [48].

6.3 Type 0A fermionic transformation

As we have argued above the type 0A string at radius $R = 1$ corresponds to Kodaira-Spencer theory on the Riemann surface

$$xy - \mu^2 = 0\,, \tag{6.32}$$

which is part of the non-compact Calabi-Yau

$$(uv + \mu)^2 = xy\,. \tag{6.33}$$

The Riemann surface has the topology of a sphere with two punctures. The two punctures correspond to the asymptotic regions, whose complex deformations are mapped to the in- and outgoing 0A tachyons, respectively.

By pulling the contour along the sphere we can relate the action of an operator in one asymptotic patch to the action in the second asymptotic patch. In this way we can, for instance, write the Ward identities (5.20) as fermionic bilinears [1]

$$\int_x \psi^*(x)x^n\psi(x)\,|V\rangle = -\int_y \psi^*(y)x^n(y)\psi(y)\,|V\rangle\,. \tag{6.34}$$

The state $|V\rangle$ is in the Hilbert space $\mathcal{H}^{\otimes 2}$, where $\mathcal{H}$ is the Hilbert space of a single free boson and it is the quantum state of the Calabi-Yau in the Kodaira-Spencer theory.

We now have to find the translation rule between the two patches. From the previous reduction of the holomorphic 3-form we see that x, y are not canonically conjugate, in contrast to the cases discussed in [1]. Instead, the 2-form (6.26) would rather suggest a commutator $[x, y] = \sqrt{xy}$, which gives rise to quantum ambiguities due to operator ordering. We will fix the ambiguities by writing x, y as operators on the Riemann surface, and requiring that they reproduce the equation of the Riemann surface (6.32) in the classical limit. In other words we will go, say, to the y-patch, put a brane at a fixed value y_0 and measure the corresponding x value, which in the classical limit is fixed by the equation of the Riemann surface (6.32).

According to the proposal of [1], the non-compact branes fibered over the Riemann surface are described by fermions $\psi(z)$, which can be bosonized as $\psi(z) = e^{-i\phi(z)/g_s}$, where $\phi(z)$ is the Kodaira-Spencer field, with $\partial\phi$ being the effective one form on the Riemann surface, which in our case in the classical limit is given by

$$\partial\phi(z)|_{\text{class.}} = \frac{\mu}{z}\,. \tag{6.35}$$

Consequently, acting with the operator $\hat{x}$ on the classical part of the fermion/brane at a fixed point y in the y-patch has to give

$$\langle\psi(y)|\,\hat{x}(y)\,|\psi(y)\rangle\,|_{\text{class.}} \to e^{i\phi(y)/g_s}\hat{x}(y)e^{-i\phi(y)/g_s} = x(y) = \frac{\mu^2}{y}\,, \tag{6.36}$$

where $\phi_{\text{class.}}(y) = \mu \log y$.

The operator that satisfies (6.36) is given by

$$\hat{x}(y) = -g_s^2 \left[\partial_y + y\partial_y^2\right] . \tag{6.37}$$

We want to note that the above is a solution to the string equation of the 0A string for $q = 0$ found in [33]

$$[L_+, L_-] = 2iM \, , \qquad \{L_+, L_-\} = 2M^2 - \frac{1}{2} \, , \tag{6.38}$$

where $L_\pm$ are the Lax operators and M is the Orlov-Shulman operator. To see this we have to identify $L_+ \to \hat{x}, L_- \to y$ and $M = \mu = H$, in the unperturbed case, where H denotes the hamiltonian. In addition we can also read this operator from the Ward identities (5.20) which are equivalent to (6.34).

We can use the representation of the operator $\hat{x}$ in the y-patch to derive the canonical transformation which transforms the branes/fermions from one patch to the other. The transformation rule is quite generally

$$\psi(y) = \int dx F(x,y)\psi(x) \, . \tag{6.39}$$

Eq. (6.37) then leads to the following differential equation for the kernel $F(x, y)$

$$\left(\partial_y + y\partial_y^2\right) F(x,y) = \frac{x}{g_s^2} F(x,y) \, . \tag{6.40}$$

Changing variables to $z = 2\frac{\sqrt{xy}}{g_s}$ we find

$$z\partial_z^2 F(z) + \partial_z F(z) = zF(z) \, , \tag{6.41}$$

which is solved by the modified Bessel function of the second kind

$$F(z) = K_0(z) = K_0\left(2\frac{\sqrt{xy}}{g_s}\right) . \tag{6.42}$$

Note that (6.42) has the integral representation

$$\int_0^\infty dz e^{-2\frac{\sqrt{xy}}{g_s}\cosh z} \, , \tag{6.43}$$

which can be rewritten as

$$\int_0^\infty \frac{dz}{z} e^{-\frac{1}{g_s}(1/z + xyz)} \, . \tag{6.44}$$

With this transformation at hand we can employ the strategy of [1], which we will discuss in section 6.6, to solve the corresponding B-model on (6.21) leading to the expressions (5.45), (5.50).

In the next section we will use the brane/fermion picture to derive the 0A reflection coefficients using the above transformation properties of the fermions.

6.4 Type 0A reflection coefficients

In this section we will show that we can derive the reflection coefficients for the 0A strings at $R = 1$ from the topological brane picture following a similar derivation for the $c = 1$ string at self-dual radius in [1]. We will set $g_s = 1$ in this section in order to match the matrix model conventions.

For the derivation we will parameterize the Riemann surface by the coordinates x, y, which, as we have seen from the reduction of the holomorphic 3-form, are not canonically conjugated. We have found that for the case $q = 0$ the classical part of the KS field in the x-patch is given by

$$\partial\phi(x) = \frac{\mu}{x}\,, \tag{6.45}$$

such that the classical piece of the fermions/branes is given by

$$\psi_{cl}(x) = e^{-i\phi_{cl}} = x^{-i\mu}\,. \tag{6.46}$$

If we evaluate the brane two point function in the x-patch we get

$$\langle 0|\, \psi(\tilde{x})\psi^*(x)\, |V\rangle = \frac{\tilde{x}^{-i\mu}x^{i\mu}}{x-\tilde{x}}\,. \tag{6.47}$$

On the other hand the correlator of two branes in the two respective asymptotic regions is given by

$$\langle 0|\, \tilde{\psi}(y)\psi^*(x)\, |V\rangle = (xy)^{i\mu} \sum_{n\geq 0} (xy)^{-n-1} R_{n+1/2}\,, \tag{6.48}$$

where $R_{n+1/2}$ is the reflection coefficient and where the classical piece of the KS field in the y-patch reads

$$\partial\phi(y) = -\frac{\mu}{y}\,, \tag{6.49}$$

which follows from the reduction of the 3-form.

By translating expression (6.47) to the y-patch using the transformation (6.39) and comparing to (6.48) we can derive the reflection coefficients. We have

$$\langle 0|\, \tilde{\psi}(y)\psi^*(x)\, |V\rangle = \int d\tilde{x} \int dz e^{-\log z - \frac{1}{z} - \tilde{x}yz} \langle 0|\, \psi(\tilde{x})\psi^*(x)\, |V\rangle\,. \tag{6.50}$$

Inserting (6.47) and comparing to (6.48) we get

$$R_{n+1/2} = \int dz e^{-\log z - \frac{1}{z}} \int dx e^{-xz} x^{n-i\mu}\,. \tag{6.51}$$

Evaluating the integrals on the positive real axis leads to the correct expression for the perturbative reflection coefficient (5.4)

$$R_{n+1/2} \sim \Gamma\left(n - i\mu + 1\right)\Gamma\left(n - i\mu + 1\right). \tag{6.52}$$

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6.5 Ramond-Ramond charge q

In the following we will point out some intriguing features of the model with $q \neq 0$.

Following the above analysis we would in this case expect the integrable structure of the 0A string to describe the deformations of the Riemann surface

$$H(x,y) = xy - \mu^2 - q^2/4 = 0\,. \tag{6.53}$$

The holomorphic 3-form can be written as

$$\Omega = \frac{1}{2}\frac{dx \wedge dy}{\sqrt{xy - q^2/4}} \wedge \frac{dv}{v}\,. \tag{6.54}$$

If we now do the integral over the 3-cycle we encounter a branch cut corresponding to the two 3-cycles whose mutual separation is determined by the parameter q. Depending on the choice of the sheet the integral $\int \Omega$ over the compact 3-cycles gives the deformation parameters $\mu \pm iq/2$. However, this implies that we cannot simply reduce the integral of Ω over the (non)compact 3-cycles to a 1-form on the Riemann surface (6.53). In addition, we have a contribution from the singular Riemann surface $xy = 0$. It would be interesting to understand this feature better by investigating in detail the disk partition function for the non-compact branes fibered over the Riemann surfaces [57].

When we take into account the second boundary contribution by using effective branes/ fermions on the Riemann surface with disk partition function $\phi_{\text{class.}}(y) = (\mu + iq/2)\log y$ we can run the machinery of the $q = 0$ case. In particular, we can use the argument in (6.36) for the Riemann surface (6.53) to find the operator

$$\hat{x}(y) = -g_s^2\left[(1+q)\partial_y + y\partial_y^2\right]. \tag{6.55}$$

We have rescaled q by a factor of g_s, $\phi_{\text{class.}}(y) = (\mu + iqg_s/2)\log y$. This dependence on g_s may hint that while μ is related to the complex deformation due to the backreaction of $N \sim 1/g_s$ compact branes, q is rather related to the non-compact branes in some novel sense.

As for the case $q = 0$, the above is a solution to the string equation of the 0A string [33], this time with finite q,

$$[L_+, L_-] = 2iM\,, \qquad \{L_+, L_-\} = 2M^2 + \frac{q^2-1}{2}\,. \tag{6.56}$$

Indeed, (6.55) is also the operator that we read from the Ward identities (5.20).

6.6 B-branes and Kontsevich-like matrix models

In the following we will use [1] to interpret the Kontsevich-like matrix model, derived in section 5, as the theory on the worldvolume of non-compact B-branes. We consider the case $q = 0$.

Consider a Calabi-Yau space of the form (6.1), as a $\mathbf{C}$ fibration over a two-dimensional complex plane spanned by x, y. The non-compact B-branes wrap the fiber directions and intersect the Riemann surface $H(x,y) = 0$ at points x_i. Denote

$$\Lambda = \mathrm{diag}(x_1, \ldots, x_N)\,. \tag{6.57}$$

The non-compact branes backreact on the geometry and deform the complex structure with deformation parameters t_n given by

$$t_n = \frac{g_s}{n} \operatorname{tr} \Lambda^{-n} \,. \tag{6.58}$$

This can be seen by realizing the non-compact branes at points x_i by fermions (6.5) and computing the one-point function of the Kodaira-Spencer field $\phi(x)$ in the background of the fermions.

We consider the two patches $x = \infty$ and $y = \infty$. The deformations of the complex structure in the x and y patch are denoted by $\bar{t}_n$ and t_n respectively, and are the couplings to positive and negative momentum tachyons. Let us set $t_n = 0$, which implies that $\partial/\partial\bar{t}_n = 0$, since all the tachyon scattering amplitudes vanish by momentum conservation. Thus, the Kodaira-Spencer field takes the form

$$y = \partial\phi(x) = \mu x^{-1} - \sum_{n>0} n\bar{t}_n x^{n-1} \,. \tag{6.59}$$

In order to deform the complex structure in the y patch, we describe the non-compact B-branes in the x patch and translate them to the y patch by using the canonical transform with the fermions transforming from patch to patch by (6.6).

In the x patch the non-compact B-branes can be described by $\psi(x) = e^{-i\phi(x)/g_s}$. Their N-point function can be evaluated by using (6.59) as

$$\langle \psi(x_1) \cdots \psi(x_N) \rangle = \Delta(x)\, e^{-\sum \phi(x_i)} = \Delta(x)\, e^{\sum_i \left(-\frac{i\mu}{g_s} \log x_i + \frac{i}{g_s} \sum_n \bar{t}_n x_i^n \right)} \,, \tag{6.60}$$

where $\Delta(x) = \prod_{i<j}(x_i - x_j)$. By performing the canonical transform (6.39) with (6.44), we get the Kontsevich-like integral of section 5.

7. Quiver gauge theories and matrix models

We will consider D-branes wrapping holomorphic cycles in Calabi-Yau 3-folds. In general, integrating out the open string degrees of freedom results in a deformed geometry, which is interpreted as the backreaction of the D-branes on the original geometry. In the topological string framework the back reaction affects the closed strings by changing the periods of the holomorphic 3-form: The Calabi-Yau space undergoes a transition where holomorphic cycles shrink and 3-cycles are opened up. In the language of four-dimensional $\mathcal{N} = 1$ supersymmetric gauge theory, the D-branes wrapping in the resolved geometry provide the UV description, while the IR physics is described by the deformed geometry after the transition.

For confining gauge theories, one assumes that the relevant IR degrees of freedom are the glueball superfields S_i. In the deformed geometry, the glueball superfields are related to the integrals of Ω by

$$S_i = \int_{\text{3-cycle}} \Omega \,. \tag{7.1}$$

The partition function of the topological field theory, as a function of the deformation parameters, computes the holomorphic F-terms of the gauge theory as a function of the glueball superfields. This has also a matrix model description [58, 59, 60, 2]. One can extend the discussion to the ring of chiral operators of the supersymmetric gauge theory and its correlators, and relate it to the matrix model and topological field theory pictures.

The whole picture can sometimes be realized by a non-critical string theory. In the case of the bosonic $c = 1$ string at the self-dual radius, the corresponding quiver gauge theory and matrix model are described by the $\hat{A}_1$ diagram. In this section we will construct the quiver gauge theories and matrix model pictures of the $\hat{c} = 1$ strings at the radii (2.7).

7.1 Super affine 0A

We will start by reviewing several aspects of the relevant resolved and deformed Calabi-Yau geometries. We will then construct the quiver gauge theory and matrix model corresponding to the superaffine 0A strings.

7.1.1 The geometry

We have seen in section 3 that the ground ring of the super-affine 0A string at the self-dual radius is a Z_2 quotient of the conifold (3.30). We will need some details on the resolution and deformation of this singular space. In this case, it is convenient to use toric variety techniques.

Here, the toric data is a set of two-vectors,

$$w_1 = (1,0)\,, \qquad w_2 = (-1,0)\,, \qquad w_3 = (0,1)\,, \qquad w_4 = (0,-1)\,, \qquad w_5 = (0,0)\,. \quad (7.2)$$

As necessary for this class of (Gorenstein canonical) singularities, these two-vectors lie in the interior or boundary of a polygon in $\mathbb{R}^2$.

From the vectors (7.2) one computes integer charges Q_l^i satisfying,

$$\sum_{i=1}^{k} Q_l^i w_i = 0\,, \qquad \sum_{i=1}^{k} Q_l^i = 0\,. \quad (7.3)$$

In our case we have

$$\begin{array}{ccc} & & \begin{array}{ccccc} z_1 & z_2 & z_3 & z_4 & z_5 \end{array} \\ \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} & = & \begin{pmatrix} 1 & 1 & 0 & 0 & -2 \\ 0 & 0 & 1 & 1 & -2 \end{pmatrix}. \end{array} \quad \begin{array}{c} (7.4) \\ (7.5) \end{array}$$

These charges represent a $\mathrm{U}(1)^2$ action on the coordinates $z_i \in \mathbb{C}^5$. The singular space is the symplectic quotient of $\mathbb{C}^5$ by $\mathrm{U}(1)^2$ defined by the (D-term) equations

$$\begin{aligned} |z_1|^2 + |z_2|^2 - 2|z_5|^2 &= 0\,, \\ |z_3|^2 + |z_4|^2 - 2|z_5|^2 &= 0\,, \end{aligned} \quad (7.6)$$

modulo the $\mathrm{U}(1)^2$ action. The resolved space is obtained by introducing two (FI) parameters t_1, t_2 , such that

$$\begin{aligned} |z_1|^2 + |z_2|^2 - 2|z_5|^2 &= t_1\,, \\ |z_3|^2 + |z_4|^2 - 2|z_5|^2 &= t_2\,. \end{aligned} \quad (7.7)$$

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We can use the second equation of (7.7) to eliminate the coordinate z_5 up to a phase factor, which can be absorbed by the second U(1). We are left with a residual Z_2 symmetry which leaves z_5 invariant, and acts as $(z_3, z_4) \to -(z_3, z_4)$. Thus, we can view the space as a Z_2 orbifold of the resolved conifold

$$\begin{aligned}
&|z_1|^2 + |z_2|^2 - |z_3|^2 - |z_4|^2 = t\,, \qquad t = t_1 - t_2\,, \\
&Z_2 : (z_3, z_4) \to -(z_3, z_4)\,,
\end{aligned} \tag{7.8}$$

where the first equation describes the resolved conifold. The resolved conifold has the structure $O(-1) \oplus O(-1) \to \mathbb{P}^1$. For $t > 0$, the Z_2 acts on the fiber leaving the $\mathbb{P}^1$ fixed. It can be viewed also as an A_1 singularity fibered over a $\mathbb{P}^1$ with volume $t_1 - t_2$. This description is valid in the semi classical regime of the Kähler moduli space $t_1, t_2 \to -\infty$ with $t_1 - t_2$ finite.

There is another semi classical regime of interest defined by $t_1, t_2 \to \infty$. In this regime equations (7.7) describe a bundle over the Hirzebruch surface $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$ with sizes t_1 and t_2 of the two $\mathbb{P}^1$'s.

In order to see that indeed the space we are considering is the 3-fold described by the ground ring of the super-affine 0A string at the self-dual radius we will review now an alternative description of the toric singularity. This is a description in terms of a collection of polynomial equations, written in terms of $\mathrm{U}(1)^2$-invariant monomials of the coordinates z_i. The monomials are generated by the nine elements

$$B_{ij} = \begin{pmatrix} z_1^2 z_3^2 z_5 & z_1^2 z_3 z_4 z_5 & z_1^2 z_4^2 z_5 \\ z_1 z_2 z_3^2 z_5 & z_1 z_2 z_3 z_4 z_5 & z_1 z_2 z_4^2 z_5 \\ z_2^2 z_3^2 z_5 & z_2^2 z_3 z_4 z_5 & z_2^2 z_4^2 z_5 \end{pmatrix}. \tag{7.9}$$

This matrix can be identified with the matrix b_{ij} in (3.11). The monomials B_{ij} satisfy the same relations as b_{ij}. For instance,

$$B_{11}B_{22} = (z_1^2 z_3^2 z_5)\,(z_1 z_2 z_3 z_4 z_5) = (z_1^2 z_3 z_4 z_5)(z_1 z_2 z_3^2 z_5) = B_{12}B_{21}\,. \tag{7.10}$$

We can identify in the toric picture the $sl(2)_L \times sl(2)_R$ affine Lie algebra of super affine 0A string theory. The $sl(2)_L$ algebra is represented by the differential operators

$$L_0 = z_1\partial_1 - z_2\partial_1\,, \qquad L_- = z_1\partial_2\,, \qquad L_+ = z_2\partial_1\,. \tag{7.11}$$

The $sl(2)_R$ is realized by similar operators with the replacement $(1 \to 3, 2 \to 4)$.

The relations among the B_{ij} can be organized in $sl(2)_L \times sl(2)_R$ representations. The unique $sl(2)_L \times sl(2)_R$ invariant deformation is given by changing only the singlet relation to

$$2B_{22}B_{22} - 2(B_{32}B_{12} + B_{23}B_{21}) + B_{31}B_{13} + B_{11}B_{33} = \mu^2\,. \tag{7.12}$$

Then the deformed space has the structure $T^*(S^3/Z_2)$, where the Z_2 acts freely on S^3.

7.1.2 Quiver gauge theories

The UV description of the quiver gauge theories is found by D-branes wrapping in the resolved geometry. Consider the semi classical regime, where the resolved geometry is described by the Z_2 quotient of the resolved conifold (7.8). As a starting point consider N_0 D3-branes placed at the singularity $t = 0$. This system can be analysed by taking $2N_0$ D3-branes [61] at the conifold and performing the Z_2 quotient [62]. One gets [63] the gauge group $\mathrm{SU}(N_0)^4$ and matter fields in bifundamental representations.

One can add fractional N_0 D3-branes by wrapping N_1 D5-branes on the resolved $\mathbb{P}^1$ in (7.8), or by wrapping $2N_1$ D5-branes on the resolved conifold [64, 65] and performing the Z_2 quotient. To see how the quotient works, let us start with the $\hat{A}_1$ quiver gauge theory with the matter content

$$\begin{array}{ccc} & \mathrm{SU}(2N_0) & \mathrm{SU}(2N_0+2N_1) \\ \mathbf{A}_i & \square & \bar{\square} \\ \mathbf{B}_j & \bar{\square} & \square \\ \mathbf{\Phi}^+ & \mathbf{adj}. & \\ \mathbf{\Phi}^- & & \mathbf{adj}. \end{array} \tag{7.13}$$

and the superpotential given by

$$W = \frac{1}{2}\left(\mathrm{tr}\ \mathbf{\Phi}^{+2} - \mathrm{tr}\ \mathbf{\Phi}^{-2}\right) - \mathrm{tr}\left(\mathbf{A}_i\mathbf{\Phi}^+\mathbf{B}_i\right) - \mathrm{tr}\left(\mathbf{A}_i\mathbf{\Phi}^-\mathbf{B}_i\right). \tag{7.14}$$

The quotient acts on the fields as

$$\mathbf{A}_i \to \mathbf{A}_i\,, \qquad \mathbf{B}_i \to -\mathbf{B}_i\,, \qquad \mathbf{\Phi}_\pm \to -\mathbf{\Phi}_\pm\,, \tag{7.15}$$

which is accompanied by the action on the Chan-Paton factors as

$$\gamma_{2N_0} = \mathrm{diag}(I_{N_0}, -I_{N_0})\,, \qquad \gamma_{2N_0+2N_1} = \mathrm{diag}(I_{N_0+N_1}, -I_{N_0+N_1})\,. \tag{7.16}$$

Note that the $\mathbb{Z}_2$ is a symmetry of the $\hat{A}_1$ theory. The quotient thus takes the form

$$\begin{aligned} \gamma_{2N_0}\mathbf{A}_i\gamma_{2N_0+2N_1} &= +\mathbf{A}_i\,, & \gamma_{2N_0+2N_1}\mathbf{B}_i\gamma_{2N_0} &= -\mathbf{B}_i\,, \\ \gamma_{2N_0}\mathbf{\Phi}^+\gamma_{2N_0} &= -\mathbf{\Phi}^+\,, & \gamma_{2N_0+2N_1}\mathbf{\Phi}^-\gamma_{2N_0+2N_1} &= -\mathbf{\Phi}^-\,. \end{aligned} \tag{7.17}$$

The matter content of the quotient of the $\hat{A}_1$ quiver theory becomes

$$\begin{array}{ccccc} & \mathrm{SU}(N_0) & \mathrm{SU}(N_0) & \mathrm{SU}(N_0+N_1) & \mathrm{SU}(N_0+N_1) \\ A_i & \square & & \bar{\square} & \\ \tilde{A}_i & & \square & & \bar{\square} \\ B_j & \bar{\square} & & & \square \\ \tilde{B}_j & & \bar{\square} & \square & \\ \Phi_1^+ & \square & \bar{\square} & & \\ \Phi_2^+ & \bar{\square} & \square & & \\ \Phi_1^- & & & \square & \bar{\square} \\ \Phi_2^- & & & \bar{\square} & \square \end{array} \tag{7.18}$$

with $i = 1, 2$. $A_i, \tilde{A}_i$ and $B_j, \tilde{B}_j$ transform as doublets under $sl(2)_L$ and $sl(2)_R$ respectively. The superpotential can be obtained from (7.14) by restricting to the fields which survive the projection

$$\begin{aligned} W_{\text{tree}} &= W_0 + W_1 \\ W_0 &= m\, tr(\Phi_1^+ \Phi_2^+) - m\, tr(\Phi_1^- \Phi_2^-)\,, \\ W_1 &= -tr(\tilde{A}_i \Phi_1^+ \tilde{B}_i) - tr(A_i \Phi_2^+ B_i) - tr(\tilde{B}_i \Phi_1^- A_i) - tr(B_i \Phi_2^- \tilde{A}_i)\,. \end{aligned} \tag{7.19}$$

Integrating out the massive fields one gets the quartic superpotential

$$W = \varepsilon^{ik} \varepsilon^{jl}\, tr(A_j \tilde{B}_k \tilde{A}_l B_i)\,. \tag{7.20}$$

As in the original $\hat{A}_1$ quiver gauge theory, the quotient gauge theory exhibits RG cascades. Suppose that the $\mathrm{SU}(N_0 + N_1) \times \mathrm{SU}(N_0 + N_1)$ gauge groups are strongly coupled. One can apply a Seiberg duality for the two gauge groups simultaneously and find a weakly coupled description. The dual theory is

$$\begin{array}{ccccc} & \mathrm{SU}(N_0 - N_1) & \mathrm{SU}(N_0 - N_1) & \mathrm{SU}(N_0) & \mathrm{SU}(N_0) \\ a_i & \square & & \bar{\square} & \\ \tilde{a}_i & & \square & & \bar{\square} \\ b_j & \bar{\square} & & & \square \\ \tilde{b}_j & & \bar{\square} & \square & \\ M_{ij} & & & \square & \bar{\square} \\ \tilde{M}_{ij} & & & \bar{\square} & \square \end{array} \tag{7.21}$$

with the superpotential

$$W = \varepsilon^{ik} \varepsilon^{jl}\, \mathrm{tr}(M_{ik} \tilde{M}_{jl}) + \mathrm{tr}\left(a_i M_{ij} b_j + \tilde{a}_i \tilde{M}_{ij} \tilde{b}_j\right)\,. \tag{7.22}$$

Here

$$M_{ij} = A_i \tilde{B}_j\,, \qquad \tilde{M}_{ij} = \tilde{A}_i B_j\,, \tag{7.23}$$

and $a_i, \tilde{a}_i, b_j, \tilde{b}_j$ are dual quarks. By integrating out the meson fields, one gets the superpotential

$$W = \varepsilon^{ik} \varepsilon^{jl}\ \mathrm{tr}(a_i \tilde{b}_j \tilde{a}_k b_l)\,. \tag{7.24}$$

This is identical to (7.20). The gauge groups and matter content are encoded in the quiver diagram, see figure 1.

Let us now discuss the classical moduli space of vacua of the quiver theory given in figure 1, which is parametrized by the invariant traces of the chiral operators. Contrary to the notation in (7.21), the bifundamental fields $a, \tilde{a}, b, \tilde{b}$ will be denoted by the capital letters $A, \tilde{A}, B, \tilde{B}$ from now on. The generators of the chiral ring are quartic expressions in the fields of the form

$$O_{i_1 j_1 i_2 j_2} = A_{i_1} \tilde{B}_{j_1} \tilde{A}_{i_2} B_{j_2}\,. \tag{7.25}$$

These operators are symmetric in the indices i_l as well as in the indices j_l, as can be seen

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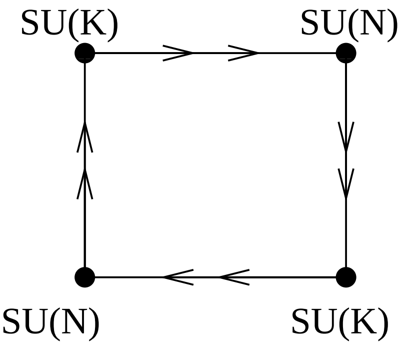

Figure 1: Quiver gauge theory for the A_1 fibered over $\mathbf{P}^1$ geometry, $N = N_0$, $K = N_0 - N_1$.

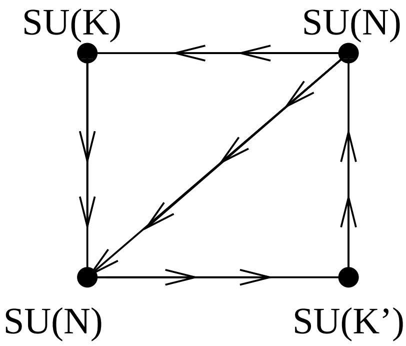

Figure 2: Quiver gauge theory for the bundle over $\mathbf{P}^1 \times \mathbf{P}^1$ geometry $N = N_0$, $K = N_0 - N_1$, $K' = N_0 + N_1$.

from the field equations of (7.42),

$$A_{i_1}\tilde{B}_{j_1}\tilde{A}_{i_2} = A_{i_2}\tilde{B}_{j_1}\tilde{A}_{i_1}\,, \qquad \tilde{B}_{j_1}\tilde{A}_{i_2}B_{j_2} = \tilde{B}_{j_2}\tilde{A}_{i_2}B_{j_1}\,. \tag{7.26}$$

$A_{i_1}\tilde{B}_{j_1}\tilde{A}_{i_2}B_{j_2}$ can be matched with the super-affine 0A ground ring generators b_{ij} (3.11), by noting that one can write the quadratic expressions (7.26) as $a_{i_1j_1}a_{i_2j_2}$ subject to the relation (3.12). One can verify the ring relations by using the field equations. For instance,

$$b_{11}b_{22} = O_{1111}O_{1122} = O_{1112}O_{1121} = b_{12}b_{21}\,. \tag{7.27}$$

Let us consider now the IR dynamics of the quiver gauge theory. Of particular interest for us is the case of N_0 being an integer multiple of N_1. In this case, after the duality cascades, one ends up with a confining gauge theory with the gauge group $\mathrm{SU}(N_1)\times\mathrm{SU}(N_1)$ and the massive bifundamentals Φ_i^+, $i = 1, 2$. In the deformed geometry $N = N_1$ sets the size of S^3 which we identify with the glueball superfield S. The holomorphic F-terms of this theory $\mathcal{F}(S)$ as a function of S are twice that of $\mathrm{SU}(N)$ SYM and are related to the perturbative super affine 0A free energy as[8]

$$\mathcal{F}_{\mathrm{SYM}}(S) = \mathcal{F}_{\mathrm{0A}}^{\mathrm{super-aff.}}(R_{\text{self-dual}})(\mu)\,, \tag{7.28}$$

where $S \sim \mu$.

The quiver gauge theory arising from D-branes wrapping in the semi classical regime $t_1, t_2 \to \infty$ is described by figure 2 and was argued to be (toric) Seiberg dual to the quiver theory of (7.21) [66, 67].

7.1.3 Matrix model

One can also write a DV matrix integral description of the quiver gauge theory F-terms. It takes the form:

$$Z = \frac{1}{V}\int d\Phi_i^+ d\Phi_i^- dA_i dB_j d\tilde{A}_i d\tilde{B}_j exp\left[-\frac{1}{g_s}W_{\mathrm{tree}}(\Phi_i^+, \Phi_i^- A_i, B_j, \tilde{A}_i, \tilde{B}_j)\right], \tag{7.29}$$

with W_{tree} given by (7.19), and V is the volume of the groups $\mathrm{SU}(\hat{N}_0)^2 \times \mathrm{SU}(\hat{N}_0 - \hat{N}_1)^2$.

[8]Of course we have to take into account possible non-trivial relations between the free energy and the holomorphic F-terms. Recall, for instance, that the planar free energy is related to the superpotential by a derivative w.r.t. S [60].

For $N_0 = N_1 = N$ one can use the matrix integral

$$Z = \frac{1}{V(\mathrm{SU}(\hat{N}) \times \mathrm{SU}(\hat{N}))} \int d\Phi_1^+ d\Phi_2^+ e^{-\frac{m}{g_s} \mathrm{tr}(\Phi_1^+ \Phi_2^+)}, \tag{7.30}$$

which gives perturbatively

$$\mathcal{F}_{\text{matrix model}}(S) = 2 \times \mathcal{F}_{c=1}(R_{\text{self-dual}})(\mu) = \mathcal{F}_{0\mathrm{A}}^{\text{super-affine}}(R_{\text{self-dual}})(\mu), \tag{7.31}$$

with $S = g_s \hat{N}$. Thus, we see that the F-terms of the gauge theory in figure 1 with $N_0 = N_1 = N$ are related to the partition function of superaffine 0A string at the radius (2.7).

7.2 Circle line theories

We will start by reviewing several aspects of the relevant resolved and deformed Calabi-Yau geometries. We will then construct the quiver gauge theory and matrix model corresponding to the 0A (0B) strings.

7.2.1 The geometry

The $\mu = 0$ ground ring of the 0A string at $R = 1$ and of the 0B string at $R = 2$ define the same singular space. Here, the toric data is a set of two-vectors,

$$w_1 = (2,0), \quad w_2 = (2,1), \quad w_3 = (1,0), \quad w_4 = (1,1), \quad w_5 = (0,0), \quad w_6 = (0,1). \tag{7.32}$$

From the vectors (7.32) one computes integer charges Q_l^i

$$\begin{array}{cc} & \begin{array}{cccccc} z_1 & z_2 & z_3 & z_4 & z_5 & z_6 \end{array} \\ \begin{pmatrix} Q_1 \\ Q_2 \\ Q_3 \end{pmatrix} = & \begin{pmatrix} 1 & -1 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 0 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{pmatrix} \end{array}. \tag{7.33}$$

The resolved singularity is described by the equations

$$\begin{aligned} |z_1|^2 - |z_2|^2 - |z_3|^2 + |z_4|^2 &= t_1, \\ |z_2|^2 - 2|z_4|^2 + |z_6|^2 &= t_2, \\ |z_3|^2 - |z_4|^2 - |z_5|^2 + |z_6|^2 &= t_3. \end{aligned} \tag{7.34}$$

modulo the $\mathrm{U}(1)^3$ action. For zero resolution parameters t_1, t_2, t_3 these equations define the toric singularity of 0A at $R = 1$.

In order to see that indeed the space we are considering is the 3-fold described by the ground ring of the 0A string at the radius (2.7), we check that the ring of $\mathrm{U}(1)^3$-invariant monomials of the coordinates z_i, is generated by the four elements

$$u = z_1 z_3 z_5, \qquad v = z_2 z_4 z_6, \qquad x = z_1^2 z_2^2 z_3 z_4, \qquad y = z_3 z_4 z_5^2 z_6^2, \tag{7.35}$$

with the relation

$$(uv)^2 - xy = 0, \tag{7.36}$$

which is the ground ring relation (3.20).

Consider the semi classical regime $t_2 \to -\infty$. The second equation of (7.34) eliminates z_4 up to a phase factor, which can be fixed by using the second U(1) symmetry. This leaves a Z_2 action $(z_2, z_6) \to (-z_2, -z_6)$. Consider in addition the regime $t_1 \to -\infty, t_3 \to \infty$, with $t_1 + t_3 = t$, with t fixed. Adding the first and third equations (7.34) we get

$$|z_1|^2 + |z_6|^2 - |z_2|^2 - |z_5|^2 = t\,, \tag{7.37}$$

$$Z_2 :\ (z_2, z_6) \to (-z_2, -z_6)\,. \tag{7.38}$$

This is the the resolved conifold $O(-1) \oplus O(-1) \to \mathbb{P}^1$, and the Z_2 acts on both the fiber and the $\mathbb{P}^1$.

7.2.2 Type 0A quiver gauge theories

One can construct the quiver gauge theory by wrapping branes on the resolved singularity in the semi classical regime discussed above, as in [68]. Alternatively, we can start from the $\hat{A}_1$ quiver gauge theory (7.13) and take a Z_2 quotient defined by

$$\mathbb{Z}_2 : \qquad \mathbf{A}_2 \to -\mathbf{A}_2, \quad \mathbf{B}_1 \to -\mathbf{B}_1, \quad \mathbf{\Phi}^{\pm} \to -\mathbf{\Phi}^{\pm}\,, \tag{7.39}$$

together with the action on the Chan-Paton factors. One can see that the matter content is given by

	$\mathrm{SU}(N_0+N_1)$	$\mathrm{SU}(N_0)$	$\mathrm{SU}(N_0+N_1)$	$\mathrm{SU}(N_0)$
$\tilde{A}_1$	$\bar{\square}$	$\square$		
$\tilde{B}_2$	$\square$	$\bar{\square}$		
$\tilde{A}_2$		$\square$	$\bar{\square}$	
$\tilde{B}_1$		$\bar{\square}$	$\square$	
A_1			$\bar{\square}$	$\square$
B_2			$\square$	$\bar{\square}$
A_2	$\bar{\square}$			$\square$
B_1	$\square$			$\bar{\square}$
Φ_1^+	$\square$		$\bar{\square}$	
Φ_2^+	$\bar{\square}$		$\square$	
Φ_1^-		$\square$		$\bar{\square}$
Φ_2^-		$\bar{\square}$		$\square$

(7.40)

Note that the $sl(2)_L \times sl(2)_R$ symmetry is broken.

The superpotential reads

$$\begin{aligned} W_{\text{tree}} &= W_0 + W_1\,, \\ W_0 &= m\left[\mathrm{tr}(\Phi_1^+\Phi_2^+) - \mathrm{tr}(\Phi_1^-\Phi_2^-)\right], \\ W_1 &= -\mathrm{tr}(A_1\Phi_2^+B_1) - \mathrm{tr}(A_2\Phi_1^+B_2) - \mathrm{tr}(\tilde{A}_1\Phi_1^+\tilde{B}_1) - \mathrm{tr}(\tilde{A}_2\Phi_2^+\tilde{B}_2) - \\ &\quad - \mathrm{tr}(B_1\Phi_2^-\tilde{A}_1) - \mathrm{tr}(B_2\Phi_2^-\tilde{A}_2) - \mathrm{tr}(\tilde{B}_1\Phi_1^-A_1) - \mathrm{tr}(\tilde{B}_2\Phi_1^-A_2) \end{aligned} \tag{7.41}$$

Integrating out the massive fields one gets

$$W \sim \mathrm{tr}(A_1B_2A_2B_1) + \mathrm{tr}(\tilde{A}_1\tilde{B}_2\tilde{A}_2\tilde{B}_1) - \mathrm{tr}(A_1\tilde{B}_1\tilde{A}_2B_2) - \mathrm{tr}(\tilde{A}_1B_1A_2\tilde{B}_2)\,. \tag{7.42}$$

The information of the gauge group and matter content is encoded in the quiver diagram, see figure 3.

The geometry (7.36) is the classical moduli space of vacua of the $D3$ branes. It is parametrized by the invariant traces of the projection of operators of the form

$$O_{i_1 j_1 i_2 j_2} = A_{i_1} B_{j_1} A_{i_2} B_{j_2} \,. \tag{7.43}$$

On the D3-brane branch these operators are generated by

$$\begin{aligned} a_{12} &= \tilde{A}_1 \tilde{B}_2 \,, & a_{21} &= \tilde{A}_2 \tilde{B}_1 \,, \\ b_{11} &= \tilde{A}_1 B_1 A_1 \tilde{B}_1 \,, & b_{33} &= \tilde{A}_2 B_2 A_2 \tilde{B}_2 \,. \end{aligned} \tag{7.44}$$

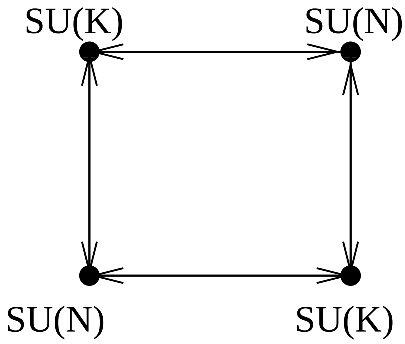

Figure 3: Quiver gauge theory for the 0A circle line theory at $R = 1$, $N = N_0$, $K = N_0 + N_1$.

The classical field equations read

$$B_2 A_2 B_1 = \tilde{B}_1 \tilde{A}_2 B_2 \,, \quad \tilde{B}_2 \tilde{A}_2 \tilde{B}_1 = B_1 A_2 \tilde{B}_2 \,, \quad B_1 A_1 B_2 = \tilde{B}_2 \tilde{A}_1 B_1 \,, \quad \tilde{B}_1 \tilde{A}_1 \tilde{B}_2 = B_2 A_1 \tilde{B}_1 \,, \tag{7.45}$$

and

$$A_1 B_2 A_2 = A_2 \tilde{B}_2 \tilde{A}_1 \,, \quad A_2 B_1 A_1 = A_1 \tilde{B}_1 \tilde{A}_2 \,, \quad \tilde{A}_1 \tilde{B}_2 \tilde{A}_2 = \tilde{A}_2 B_2 A_1 \,, \quad \tilde{A}_2 \tilde{B}_1 \tilde{A}_1 = \tilde{A}_1 B_1 A_2 \,. \tag{7.46}$$

One can verify the ring relations by using these equations,

$$(a_{12} a_{21})^2 = (\tilde{A}_1 \tilde{B}_2 \tilde{A}_2 \tilde{B}_1)^2 = (\tilde{A}_1 B_1 A_1 \tilde{B}_1 \tilde{A}_2 B_2 A_2 \tilde{B}_2) = b_{11} b_{33} \,. \tag{7.47}$$

Let us consider now the IR dynamics of the quiver gauge theory. This depends on the numbers N_0 and N_1, and is subject to Seiberg-like dualities. A particular simple case is the choice $N_0 = 0$ and $N_1 = N$, which we will consider first. In this case the gauge group is $\mathrm{SU}(N) \times \mathrm{SU}(N)$ with no massless matter and with massive bifundamentals Φ_i^+, $i = 1, 2$. This is the same theory as the one corresponding to the super affine theory. In the deformed geometry N sets the size of S^3 which we identify with the glueball superfield S. The holomorphic F-terms of this theory $\mathcal{F}(\mathcal{S})$ as a function of S are twice that of $\mathrm{SU}(N)$ SYM and are related to the perturbative type 0A free energy as

$$\mathcal{F}_{\mathrm{SYM}}(S) = \mathcal{F}_{0\mathrm{A}}(R = 1)(\mu) \,, \tag{7.48}$$

where $S \sim \mu$.

For different N_0, N_1, some of the gauge couplings become strong and one has to run a cascade of Seiberg dualities in order to study the IR physics. This can be analysed in detail, but will not be done here. As we noted before, we expect the quivers corresponding to type 0A strings to have the same IR dynamics as the above.

7.2.3 Deformed ring geometry

Gluino condensation in the D5-branes worldvolume theories deforms the ring relation (3.20) and (7.36). The deformed geometry can be seen by a D3-brane probe. The probe is described by adding the U(1)s to the quiver theory. To take into account the gauge dynamics on the D5-branes worldvolume, one introduces the Affleck-Dine-Seiberg superpotential [69], and solves for the meson fields M and $\tilde{M}$

$$W_{ADS} = (N-2)\left(\frac{\Lambda^{3N-2}}{\det M}\right)^{\frac{1}{N-2}} + (N-2)\left(\frac{\tilde{\Lambda}^{3N-2}}{\det \tilde{M}}\right)^{\frac{1}{N-2}}, \tag{7.49}$$

$$W_{\text{tree}} = \frac{1}{m}\left(m_{22}\tilde{m}_{22} + m_{11}\tilde{m}_{11} - m_{12}m_{21} - \tilde{m}_{12}\tilde{m}_{21}\right), \tag{7.50}$$

with

$$M = \begin{pmatrix} \tilde{A}_2\tilde{B}_1 & \tilde{A}_2B_2 \\ A_1\tilde{B}_1 & A_1B_2 \end{pmatrix}, \qquad \tilde{M} = \begin{pmatrix} \tilde{A}_1\tilde{B}_2 & \tilde{A}_1B_1 \\ A_2\tilde{B}_2 & A_2B_1 \end{pmatrix}. \tag{7.51}$$

The equations of motion give

$$\begin{aligned} &m_{11}m_{22} - m_{12}m_{21} = \left(\Lambda^{3N-2}m^{N-2}\right)^{\frac{1}{N-1}}, \qquad \tilde{m}_{11}\tilde{m}_{22} - \tilde{m}_{12}\tilde{m}_{21} = \left(\tilde{\Lambda}^{3N-2}m^{N-2}\right)^{\frac{1}{N-1}}, \\ &m_{11} = \tilde{m}_{22}, \qquad m_{22} = \tilde{m}_{11}. \end{aligned} \tag{7.52}$$

For the gauge invariant monomials

$$a_{12} = m_{22}, \quad a_{21} = m_{11}, \qquad b_{11} = \tilde{m}_{12}m_{21}, \qquad b_{33} = m_{12}\tilde{m}_{21}, \tag{7.53}$$

one finds the deformed geometry

$$b_{11}b_{33} = (a_{12}a_{21} - \mu)(a_{12}a_{21} - \tilde{\mu}), \tag{7.54}$$

where we introduced the parameters $\mu = \left(\Lambda^{3N-2}m^{N-2}\right)^{\frac{1}{N-1}}$ and $\tilde{\mu} = \left(\tilde{\Lambda}^{3N-2}m^{N-2}\right)^{\frac{1}{N-1}}$.

7.2.4 Matrix model

One can write a DV matrix integral description of the quiver gauge theory F-terms. It takes the form:

$$Z = \frac{1}{V}\int d\Phi_i^+ d\Phi_i^- dA_i dB_j d\tilde{A}_i d\tilde{B}_j exp\left[-\frac{1}{g_s}W_{\text{tree}}(\Phi_i^+, \Phi_i^- A_i, B_j, \tilde{A}_i, \tilde{B}_j)\right], \tag{7.55}$$

with W_{tree} given by (7.41), and V is the volume of the groups $\mathrm{SU}(\hat{N}_0)^2 \times \mathrm{SU}(\hat{N}_0 + \hat{N}_1)^2$.

For $N_0 = 0, N_1 = N$ one can use the matrix integral

$$Z = \frac{1}{V(\mathrm{SU}(\hat{N}) \times \mathrm{SU}(\hat{N}))}\int d\Phi_1^+ d\Phi_2^+ e^{-\frac{m}{g_s}tr(\Phi_1^+\Phi_2^+)}, \tag{7.56}$$

which gives perturbatively

$$\mathcal{F}_{\text{matrix model}}(S) = 2 \times \mathcal{F}_{c=1}(R_{\text{self-dual}})(\mu) = \mathcal{F}_{0\mathrm{A}}(R=1)(\mu), \tag{7.57}$$

with $S = g_s\hat{N}$. Thus, we see that that the F-terms of the gauge theory with $N_0 = 0, N_1 = N$ are related to the partition function of the 0A string at the radius (2.7). The above suggests that $\mu = \tilde{\mu}$ in (7.54). Note, that the discussion in section 6 suggests that the same matrix model describes the Z_2 orbifold of the $c = 1$ bosonic string, but now $\mu \neq \tilde{\mu}$.

7.2.5 Quiver gauge theories for 0B

One repeats the same analysis for the T-dual type 0B theory. Again, the quiver gauge theory and the superpotential can be deduced from the $\hat{A}_1$ quiver theory of the conifold. Now the Z_2-action takes the form

$$\mathbb{Z}_2 : \ \mathbf{A}_2 \to -\mathbf{A}_2, \quad \mathbf{B}_2 \to -\mathbf{B}_2 \,. \tag{7.58}$$

One has

$$\begin{array}{ccccc}
 & \mathrm{SU}(N_1) & \mathrm{SU}(N_2) & \mathrm{SU}(N_3) & \mathrm{SU}(N_4) \\
\tilde{A}_1 & \overline{\Box} & \Box & & \\
\tilde{B}_1 & \Box & \overline{\Box} & & \\
\tilde{A}_2 & & \Box & \overline{\Box} & \\
\tilde{B}_2 & & \overline{\Box} & \Box & \\
A_1 & & & \overline{\Box} & \Box \\
B_1 & & & \Box & \overline{\Box} \\
A_2 & \overline{\Box} & & & \Box \\
B_2 & \Box & & & \overline{\Box} \\
\Phi_1^+ & \mathrm{adj.} & & & \\
\Phi_2^+ & & & \mathrm{adj.} & \\
\Phi_1^- & & & & \mathrm{adj.} \\
\Phi_2^- & & \mathrm{adj.} & &
\end{array} \tag{7.59}$$

As in type 0A the $sl(2)_L \times sl(2)_R$ symmetry is broken.

The superpotential reads

$$\begin{aligned}
W &= W_0 + W_1 \,, \\
W_0 &= \frac{m}{2}\left[\mathrm{tr}(\Phi_1^+)^2 + \mathrm{tr}(\Phi_2^+)^2 - \mathrm{tr}(\Phi_1^-)^2 - \mathrm{tr}(\Phi_2^-)^2\right], \\
W_1 &= -\,\mathrm{tr}(A_1\Phi_2^+B_1) - \mathrm{tr}(A_2\Phi_1^+B_2) - \mathrm{tr}(\tilde{A}_1\Phi_1^+\tilde{B}_1) - \mathrm{tr}(\tilde{A}_2\Phi_2^+\tilde{B}_2) - \\
&\quad -\,\mathrm{tr}(B_1\Phi_1^-A_1) - \mathrm{tr}(B_2\Phi_1^-A_2) - \mathrm{tr}(\tilde{B}_1\Phi_2^-\tilde{A}_1) - \mathrm{tr}(\tilde{B}_2\Phi_2^-\tilde{A}_2) \,.
\end{aligned} \tag{7.60}$$

Upon integrating out the massive matter one gets

$$\begin{aligned}
W \sim\ & \mathrm{tr}(A_1B_1A_2B_2) - \mathrm{tr}(\tilde{A}_1\tilde{B}_1\tilde{A}_2\tilde{B}_2) - \\
& - \mathrm{tr}(A_1\tilde{B}_2\tilde{A}_2B_1) - \mathrm{tr}(\tilde{A}_1B_2A_2\tilde{B}_1) \,.
\end{aligned} \tag{7.61}$$

The bifundamental fields are related to the bifundamental fields of the 0A theory by

$$B_i^{0\mathrm{B}} = \varepsilon_{ij} B_j^{0\mathrm{A}} \,. \tag{7.62}$$

One can see that the superpotentials match, once the massive matter is integrated out. This map corresponds to the T-duality of the non-critical circle line strings.

The analysis of the IR physics and the matrix model are the same as in the type 0A case.

8. Discussion

In this paper we studied the connections between four types of systems: $\hat{c} = 1$ non-critical strings, topological B-models on quotients of the conifold, matrix models in $0+1$ and $0+0$ dimensions and $\mathcal{N} = 1$ supersymmetric quiver gauge theories in four dimensions. The basic tool for identifying these different systems was the commutative and associative ring structure.

We considered two families of non-critical strings: the circle line theories 0A and 0B at particular radii, and the super affine theories at their self-dual radii. Based on the analysis of the ground ring and the quiver gauge theories we proposed these non-critical strings as equivalent (non-perturbative) descriptions of the topological B-model on the respective quotients of the conifold.

We discussed the superaffine and circle line models of the 0A string at special radii in more detail. For the superaffine theory we derived the partition function. The analysis of the 0A matrix model revealed that the τ-function of the Toda hierarchy that describes the 0A tachyon correlators can be lifted to a two matrix integral. Following [1] we attempted to identify the integrable structure of the 0A string in the B-model on the appropriate quotient of the conifold leaving us with many interesting open questions. For one thing we constructed the deformed ring relations by indirect means. It would be interesting to verify them. We considered the models at special radii. It would be interesting to analyse whether the systems have topological descriptions at multiples of these radii.[9] Another interesting direction to go would be to analyze the B-model at the points $q \neq 0, \mu \to 0$ and $q \to 0$ as outlined in section 6. In that context one should also strive for a better understanding of the role of T-duality. The tachyon perturbations of the T-dual model, or equivalently the winding mode perturbations, should match to deformations of the fiber, which was parametrized by u, v in our conventions. Also, we have included only one RR charge. It would be interesting to analyse the case with two RR charges different than zero. Another generalization that deserves consideration is the (RR background) non-critical superstring [70].

Note, that although the non-critical strings that we studied are well defined non-perturbatively, we established the connections between the different systems only perturbatively. In the B-model there are genuine non-perturbative terms corresponding to D-branes.[10] The non-perturbative terms in the matrix model should give non-perturbative graviphoton terms in the effective gauge system [25]. It seems worthwhile to understand better the relation between the non-perturbative contributions in the matrix model and in the B-model [72]. It would be interesting if further understanding can be gained from the lagrangian branes of the topological A-model on the quotients of the conifold, as the S-duality between the topological A and B models suggests [73] and to uncover the relation to other non perturbative aspects of topological strings discussed recently in [74, 75].

[9] See section 6 for a discussion of the bosonic $c = 1$ string at multiples of the self-dual radius.

[10] For a recent study of D-brane instantons in the topological B-model see [71].

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A. Leg pole factors for the 0A theory

The leg pole factors for the deformed matrix model have been derived in [41] for the case $\mu = 0$. It is no more difficult to derive them for the general case, see also [56]. As in the $c = 1$ [76, 77] case we can extract the physical tachyon operators T_p from the small l limit of the macroscopic loop operators

$$W\left(l, p\right) \sim \int dt e^{ipt} \operatorname{Tr} e^{-l\Phi(t)^2} = \Gamma\left(-p\right) l^p T_p\,, \tag{A.1}$$

see also [42]. If we consider a correlator of the loop operators

$$\left\langle \prod_i^n W\left(l_i, p_i\right) \right\rangle_c = \int \left(\prod_i d\lambda_i e^{-l_i \lambda_i} \right) \left\langle \prod_i \rho\left(\lambda_i, p_i\right) \right\rangle_c\,, \tag{A.2}$$

introduce the classical time τ through

$$\lambda^2 = \mu + \sqrt{q^2 + 4\mu^2 - \frac{1}{4}} \cosh 2\tau\,, \tag{A.3}$$

and consider the limit $\tau \to \infty$, $l_i \to 0$ in (A.1) we find that

$$\left\langle \prod_i^n W\left(l_i, p_i\right) \right\rangle_c \sim \prod_i \left(\Gamma\left(-p_i\right) l^{p_i} 2^{-p_i/2} \left(q^2 + 4\mu^2 - \frac{1}{4}\right)^{p_i/4} \right) \mathcal{R}\left(p_k\right), \tag{A.4}$$

where the part $\mathcal{R}(p_k)$ is given by products of the reflection coefficients according to the rules of [42]. If we denote by V_p the tachyon vertex operators normalized according to the conventions of this paper and by T_p the physical tachyon vertex operators we have the to use the following translation rules

$$\left\langle \prod_{i=1}^n T_{p_i} \right\rangle = \mu^{2-n} \prod_{i=1}^n \left(\frac{q^2 + 4\mu^2 - 1/4}{4} \right)^{|p_i|/4} \left\langle \prod_{i=1}^n V_{p_i} \right\rangle. \tag{A.5}$$

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